Appendix A

Relations for the ∇ operator

A.1 Proofs of the product relations in Chapter 1

To prove Eqs. (1.10) and (1.11) we will use two relations supposedly known from basic courses in physics or mathematics. Let $\boldsymbol{u}$ and $\boldsymbol{A}$ represent vector fields, and Φ a scalar field.

- *Gauss's theorem*:

$$\int_V \operatorname{div} \boldsymbol{u}\, dV = \oint_S \boldsymbol{u}\cdot d\boldsymbol{S} \tag{A.1}$$

- *Stokes's theorem*:

$$\int_S (\operatorname{curl} \boldsymbol{u})\cdot d\boldsymbol{S} = \oint_L \boldsymbol{u}\cdot d\boldsymbol{L} \tag{A.2}$$

V is a volume, S a surface, and L a curve embedded in the surface. The integral sign $\oint$ denotes a closed (og thus enveloping) surface or curve. The line element $d\boldsymbol{L}$ is a vector tangential to the curve. In the surface element $d\boldsymbol{S} = \hat{\boldsymbol{n}}\, dS$, $\hat{\boldsymbol{n}}$ is a perpendicular unit vector in a point on the surface. For a closed surface, $\hat{\boldsymbol{n}}$ points by definition out of the enveloped volume, and the direction of circulation around a closed curve corresponds to a positive rotation around the normal vector on the enclosed surface. In Eq. (A.2) it is assumed that $\operatorname{curl} \boldsymbol{u}$ is continuous.

Let now $\boldsymbol{u} = -\operatorname{grad} \Phi$ in Eq. (A.2). Then the RHS equals zero, since the integrand in it is a total differential

$$\boldsymbol{u}\cdot d\boldsymbol{L} \;=\; u_i\, dx_i \;=\; -\partial_i \Phi\, dx_i \;=\; -d\Phi$$

and thus the integral around the closed curve equals zero. Since the LHS must hold for any surface S for the arbitrary closed curve L, the integrand must satisfy $\operatorname{curl} \operatorname{grad} \Phi = 0$. Eq. (1.10) is thus proved.

Alternatively, let us then replace $\boldsymbol{u}$ with another vector field $\boldsymbol{A}$ in Eq. (A.2), and let the curve L shrink to a point in such a way that S becomes a closed surface enclosing a volume. (I the conditions for Stokes's theorem, the surface S may have the form of a "bag".) Then

$$\oint_S (\operatorname{curl} \boldsymbol{A})\cdot d\boldsymbol{S} = 0$$

because the line integral on the RHS becomes zero when the integration path shrinks to a point (provided the vector field $\boldsymbol{A}$ is continuous). If we then let $\boldsymbol{u} = \operatorname{curl} \boldsymbol{A}$ ($\boldsymbol{A}$ thus a *vector potential*) in Eq. (A.1), we get $\operatorname{div} \operatorname{curl} \boldsymbol{A} = 0$, since Gauss's theorem must hold for all choices of volume V enclosed by a surface S. Eq. (1.11) has thus been proved.

To prove the validity of Eqs (1.12), (1.13) og (1.14) we use the *contraction relation*

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{A.3}$$

between the Levi-Civitá tensor (1.9) and the Kronecker delta (1.5). Inserting, and using that the Levi-Civitá tensor does not change its value if the indices are rotated, we get:

$$\begin{aligned}(\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{u})_i &= \epsilon_{ijk}\partial_j\epsilon_{kmn}\partial_m u_n \\ &= \epsilon_{kij}\epsilon_{kmn}\partial_j\partial_m u_n \\ &= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\partial_j\partial_m u_n \\ &= \partial_n\partial_i u_n - \partial_j\partial_j u_i \\ &= \partial_i(\mathrm{div}\,\boldsymbol{u}) - \nabla^2 u_i \end{aligned} \tag{A.4}$$

$$\begin{aligned}(\boldsymbol{u}\times\mathrm{curl}\,\boldsymbol{u})_i &= \epsilon_{ijk}u_j\epsilon_{kmn}\partial_m u_n \\ &= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})u_j\partial_m u_n \\ &= u_n\partial_i u_n - u_m\partial_m u_i \\ &= \frac{1}{2}\partial_i(\boldsymbol{u}^2) - (\boldsymbol{u}\cdot\nabla)u_i \end{aligned} \tag{A.5}$$

Eqs. (1.12) and (1.13) have thus been proved. To prove (1.14) we first take the curl of the LHS of Eq. (A.5), and use the rule for derivation of a product:

$$\begin{aligned}&(\mathrm{curl}(\boldsymbol{u}\times\mathrm{curl}\,\boldsymbol{u}))_i \\ &= \epsilon_{ijk}\partial_j\epsilon_{kmn}u_m\epsilon_{nop}\partial_o u_p \\ &= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\epsilon_{nop}\partial_j u_m\partial_o u_p \\ &= \epsilon_{jop}\partial_j u_i\partial_o u_p - \epsilon_{iop}\partial_j u_j\partial_o u_p \\ &= \epsilon_{jop}(\partial_j u_i)\partial_o u_p + \epsilon_{jop}(u_i\partial_j)\partial_o u_p - \epsilon_{iop}(\partial_j u_j)\partial_o u_p - \epsilon_{iop}(u_j\partial_j)\partial_o u_p \\ &= ((\mathrm{curl}\,\boldsymbol{u})\cdot\nabla)u_i + ((\nabla\times\nabla)\cdot\boldsymbol{u})u_i - (\nabla\cdot\boldsymbol{u})(\mathrm{curl}\,\boldsymbol{u})_i - (\boldsymbol{u}\cdot\nabla)(\mathrm{curl}\,\boldsymbol{u})_i \end{aligned} \tag{A.6}$$

The term with $\nabla\times\nabla$ on the RHS is zero because of the usual rule for a vector product. However the three Levi-Civitá symbols in $\mathrm{curl}(\boldsymbol{u}\times\mathrm{curl}\,\boldsymbol{u})$ can also be contracted in an alternative order, corresponding to taking the *curl* of the RHS of Eq. (A.5):

$$\begin{aligned}(\mathrm{curl}(\boldsymbol{u}\times\mathrm{curl}\,\boldsymbol{u}))_i - (\mathrm{curl}((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}))_i &= \epsilon_{ijk}\partial_j(u_n\partial_k u_n) \\ &= \epsilon_{ijk}(\partial_j u_n)(\partial_k u_n) + \epsilon_{ijk}u_n\partial_j\partial_k u_n \\ &= 0 \end{aligned} \tag{A.7}$$

That the RHS of Eq. (A.7) equals zero, follows by letting the 'dummy' summation indices j and k interchange their names (the sum being invariant, of course). The Levi-Civitá symbol is antisymmetric in the interchange of two indices, such that

$$\epsilon_{ijk} = -\epsilon_{jik}$$

Since the rest of the factors in each product in the nest-to-last line of Eq. (A.7) are symmetric in the interchange of j and k, we find that each term must be equal to itself with the opposite sign, and thus must be zero.

Eqs. (A.6) and (A.7) taken together give Eq. (1.14), which has thus been proved. Also the relations (1.10) and (1.11) could have been shown in a simple way by the symmetry arguments used above.

Figure A.1: Cylindrical (a) og spherical coordinates (b)

A.2 Expressions in curvilinear coordinates

The following expressions refer to the quantities defined in Figure A.1. A more precise notion of the two coordinate systems we will restrict ourselves to consider, is *cylindrical* and *spherical polar coordinates*. Notice that $\hat{\boldsymbol{e}}_r$ denotes different things in the two coordinate systems.

The summary below is not assumed to be exhaustive. We will only include expressions which are needed for the exposition in these lecture notes, as well as in problem solving. Formula collections like [Rottmann 1995] give a more complete survey.

- *Cylindrical coordinates:*

$$\boldsymbol{u} = u_r\hat{\boldsymbol{e}}_r + u_\phi\hat{\boldsymbol{e}}_\phi + u_z\hat{\boldsymbol{e}}_z \quad (A.8)$$

$$\nabla = \hat{\boldsymbol{e}}_r\frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\phi\frac{1}{r}\frac{\partial}{\partial\phi} + \hat{\boldsymbol{e}}_z\frac{\partial}{\partial z} \quad (A.9)$$

$$\nabla\cdot\boldsymbol{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z} \quad (A.10)$$

$$(\nabla\times\boldsymbol{u})_z = \frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r} - \frac{1}{r}\frac{\partial u_r}{\partial\phi} \quad (A.11)$$

$$((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})_r = u_r\frac{\partial u_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial\phi} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\phi^2}{r} \quad (A.12)$$

$$(\nabla^2\boldsymbol{u})_r = \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial\phi^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial\phi} \quad (A.13)$$

$$(\nabla^2\boldsymbol{u})_\phi = \frac{\partial^2 u_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial\phi^2} + \frac{\partial^2 u_\phi}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\phi} \quad (A.14)$$

$$(\nabla^2\boldsymbol{u})_z = \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial\phi^2} + \frac{\partial^2 u_z}{\partial z^2} \quad (A.15)$$

$$dV = r\,dr\,d\phi\,dz \quad (A.16)$$

- *Spherical coordinates:*

$$\boldsymbol{u} = u_r\hat{\boldsymbol{e}}_r + u_\theta\hat{\boldsymbol{e}}_\theta + u_\phi\hat{\boldsymbol{e}}_\phi \quad (A.17)$$

$$\nabla = \hat{\boldsymbol{e}}_r\frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{e}}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \quad (A.18)$$

$$\nabla\cdot\boldsymbol{u} = \frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \operatorname{cotg}\theta\,\frac{u_\theta}{r} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} \quad (A.19)$$

$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi \quad (A.20)$$

Notice the difference between the following two expressions in the D-dimensional ($D \in \{2,3\}$) spherically symmetric case:

$$\nabla^2\Phi = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}(r^{D-1}\frac{\partial\Phi}{\partial r}) \quad \text{(spherical symmetry)} \quad (A.21)$$

$$(\nabla^2\boldsymbol{u})_r = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}(r^{D-1}\frac{\partial u_r}{\partial r}) - (D-1)\frac{u_r}{r^2} \quad \text{(spherical symmetry)} \quad (A.22)$$

It arises because Φ is a scalar, independent of the choice of coordinate system, while u_r is a vector component from an expression which had its basic definition in cartesian coordinates. For $D = 2$ the expressions may refer both to plane polar coordinates and to cylindrical coordinates with z independence. The factor r^{D-1}, coming from the div part of ∇^2, arises because the expression can be interpreted as the divergence of a volume flow velocity.