Appendix B

Derivation of the Navier-Stokes equations

The derivation is assumed to be known. We will summarize it here because the formalism is also used in the derivation of the expression for viscous energy loss in Chapter 2. It is presented for a *compressible* fluid, with eventual specialization to the incompressible case. An alternative version of the derivation of Eq. (B.13), also in tensor notation, can be found in [Papatzacos 2003].

B.1 The stress tensor

We introduce the *stress tensor* $\|\sigma\|$ (see also Eq. (**??**)), a matrix with components

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \tag{B.1}$$

where τ_{ij} contains stresses due to the viscous effects. The first index specifies which component of the force per unit area one considers, the second specifies which surface is considered by the direction of its normal vector. Some components of the tensor $\|\tau\|$ are shown in Figure B.1. The viscous tensor τ_{ij} must be *symmetric* ($\tau_{ij} = \tau_{ji}$), which follows from momentum balance for a volume element of fluid (absence of infinite rotation). The total force per unit volume in the i direction is obtained by computing the variation in the direction of the surface vector. Using the summation convention,

$$\partial_j \sigma_{ij} = -\partial_i p + \partial_j \tau_{ij} \tag{B.2}$$

We notice that Newton's 2. law for a unit volume,

$$\rho D_t u_i = \partial_j \sigma_{ij} + \rho g_i \tag{B.3}$$

reduces to the Euler equation (1.24) in the case without viscous effects.

B.2 The angular deformation velocity tensor

The viscous part of the stress tensor will depend upon the fluid properties and upon the deformation rate for a fluid element. To get a measure of the deformation rate we first

Figure B.1: The stress tensor

Figure B.2: Relative movement in the fluid

introduce the *velocity gradient tensor* ζ_{ij}:

$$\zeta_{ij} = \partial_i u_j \tag{B.4}$$

This is the rate of rotation for a line element with direction i due to the component of velocity in the j direction. The tensor can be expressed as a sum of one symmetric part, e_{ij}, and another antisymmetric part, η_{ij}:

$$\zeta_{ij} = e_{ij} + \eta_{ij} \tag{B.5}$$

$$e_{ij} = \frac{1}{2}(\zeta_{ij} + \zeta_{ji}) \tag{B.6}$$

$$\eta_{ij} = \frac{1}{2}(\zeta_{ij} - \zeta_{ji}) \tag{B.7}$$

Referring to Figure B.2 we compute the rate of strain for the difference δl between two material points A and B in the fluid:

$$(\delta l)^2 = \delta x_i \, \delta x_i \tag{B.8}$$

$$\begin{aligned} D_t(\delta l)^2 &= 2\,\delta x_i \, D_t(\delta x_i) \\ &= 2\,\delta x_i \, \delta u_i \\ &= 2\,\delta x_i \, \delta x_j \, \partial_j u_i \\ &= 2\,\delta x_i \, \delta x_j \, e_{ij} \end{aligned} \tag{B.9}$$

The rate of strain thus depends upon the components of the symmetric combination e_{ij} in Eq. (B.6), the *angular deformation velocity tensor*[1]. As can be easily checked, the combination $\zeta_{ij} + \zeta_{ji}$ measures the rate of the angular deformation at the edge ij of a cartesian volume element. Thus, the tensor $\|e\|$ measures the rate of deformation of a fluid element.

The antisymmetric combination $\|\eta\|$ in Eq. (B.7) contains no deformation contributions:

$$\begin{aligned} \eta_{ij} &= \frac{1}{2}(\partial_i u_j - \partial_j u_i) \\ &= \frac{1}{2}\epsilon_{ijk}\epsilon_{kmn}\partial_m u_n \\ &= \frac{1}{2}\epsilon_{ijk}\omega_k \end{aligned} \tag{B.10}$$

Thus, η_{ij} corresponds to the *vorticity*, Eq. (1.20).[2] It is a measure of the property of the velocity field which corresponds to rotation without deformation.

B.3 Newtonian fluid

For a *Newtonian fluid* the components of the stress tensor depend linearly on the components of the angular deformation velocity tensor:

$$\tau_{ij} = \Lambda_{ijkl} e_{kl} \tag{B.11}$$

Since physical processes are independent of the axis directions chosen, $\|\Lambda\|$ must be an *isotropical* tensor, and the most general form of such a tensor is (see also the comment at the end of Appendix D)

$$\Lambda_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \xi \delta_{ik}\delta_{jl} + \chi \delta_{il}\delta_{jk} \tag{B.12}$$

This gives

$$\begin{aligned} \tau_{ij} &= \lambda \delta_{ij} e_{kk} + (\xi + \chi) e_{ij} \\ &\equiv 2\mu e_{ij} + \lambda \delta_{ij} e_{kk} \\ &= \mu \left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\partial_k u_k\right) + \zeta \delta_{ij}\partial_k u_k \end{aligned} \tag{B.13}$$

where the coefficient

$$\zeta = \lambda + \frac{2}{3}\mu \tag{B.14}$$

[1] This tensor can also be found in the literature without the factor 1/2 included.

[2] The vorticity vector is an example of an *axial* vector, whose components are given by the components of an antisymmetric tensor, as distinct from a "usual" *(polar)* vector. More about axial vectors in Appendix D.

is defined such that the expression in the parentheses on the RHS of Eq. (B.13) equals zero for $i = j$ (with summation implied).[3] [4]

Eq. (B.13) is then inserted into Eqs. (B.1) and (B.2):

$$\partial_j \tau_{ij} = \mu((\partial_j \partial_j) u_i + \frac{1}{3}\partial_i(\partial_j u_j)) + \zeta \partial_i(\partial_j u_j) \tag{B.15}$$

From Eq. (B.3) we then get the equation of motion for a compressible Newtonian fluid:

$$\rho D_t \boldsymbol{u} = -\nabla p + \rho \boldsymbol{g} + \mu \nabla^2 \boldsymbol{u} + (\zeta + \frac{\mu}{3})\nabla(\nabla \cdot \boldsymbol{u}) \tag{B.16}$$

In the incompressible case we get Eq. (1.25), which is the Navier-Stokes equation.

The formalism in this derivation is quite analogous to the one for the static dependence between stresses and deformations in an elastic continuous solid [Sommerfeld 1964]. The existence of *two* coefficients of viscosity, μ and ζ, corresponds to the appearance of two constants, Young's modulus and the Poisson ratio, in the description of the elastic properties.

Among expressions for the components of $\|\tau\|$ in curvilinear coordinates we will present two in spherical and one in cylindrical polar coordinates (see Appendix A) for the special case of an incompressible fluid, to be used in Chapter 4:

$$\tau_{rr} = 2\mu \frac{\partial u_r}{\partial r} \quad \text{(spherical)} \tag{B.17}$$

$$\tau_{\theta r} = \mu \{\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\} \quad \text{(spherical)} \tag{B.18}$$

$$\tau_{\phi r} = \mu \{\frac{1}{r}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\} \quad \text{(cylindrical)} \tag{B.19}$$

[3]By choosing a velocity field with $e_{12} = \frac{1}{2}\partial u/\partial y$, and the other $e_{ij} = 0$, we see that the definition of μ is such that it coincides with the basic definition of the viscosity coefficient from elementary courses.

[4]The viscosity coefficient ζ is usually of the same order of magnitude as μ. For compressions or dilatations ($\rho \neq$ konstant) ζ will get contributions from relaxation processes where the substance adapts to a new thermodynamical equilibrium. Such processes are irreversible and involve a change of entropy. They take place over a characteristic period of time, the *relaxation time*. Therefore, ζ cannot be a constant, properly speaking [Landau and Lifshitz 1987]; it depends on the order of magnitude of the time scale of the fluid dynamical processes as compared to the molecular relaxation time.