Appendix C

Some thermodynamical relations

The extensive thermodynamical quantities ϵ, V, h and s for internal energy, volume, enthalpy and entropy, respectively, with

$$h = \epsilon + pV \tag{C.1}$$

will be defined per unit mass,[1] so that $\rho = 1/V$. The first law of thermodynamics can then be written in the following two ways, among others, for a reversible process [Huang 1963]:[2]

$$
\begin{aligned}
d\epsilon &= T\,ds - p\,dV \\
&= T\,ds + \frac{p}{\rho^2}\,d\rho
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
dh &= T\,ds + V\,dp \\
&= T\,ds + \frac{1}{\rho}\,dp
\end{aligned}
\tag{C.3}
$$

By expressing the differentials in Eq. (C.3) by the differential of a spatial variable, we get

$$\nabla p = \rho \nabla h - \rho T \nabla s \tag{C.4}$$

For an adiabatical process $Ds/Dt = 0$, so that

$$\partial_t s + (\boldsymbol{u} \cdot \nabla)s = 0 \tag{C.5}$$

In these lecture notes the relations above are used in Chapter 2 for the derivation of the expression for energy flow density in a compressible ideal fluid.

The velocity of sound, c, in a fluid is given by the adiabatic compressibility β_s, or equivalently by the isothermal compressibility β_T and the relation γ between the specifical heats c_p and c_v at constant pressure and constant volume, respectively:

$$
\begin{aligned}
c^2 &= \frac{1}{\rho \beta_s} \\[4pt]
\beta_{s,T} &= \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_{s,T} \\[4pt]
\beta_s &= \gamma \beta_T \\[4pt]
\gamma &= \frac{c_p}{c_v}
\end{aligned}
$$

$$\tag{C.6}$$
$$\tag{C.7}$$
$$\tag{C.8}$$
$$\tag{C.9}$$

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[1] Molar quantities are preferable in general. However, the usage per unit mass chosen here is useful for some derivations in Chapter 2.

[2] The sign of the dV term is chosen such that positive work is done *by* the system *on* its surroundings.

These relations have been used in Chapter 2 in the derivation of the range of validity of the approximation of an incompressible fluid. For an ideal one-atomic gas, $\gamma = 5/3$. For real gases in the ideal gas limit ($p \rightarrow 0$), γ is in general a function of T; for air (a two-atomic gas) at standard conditions, $\gamma \approx 7/5$.

The equation of state for an adiabatic ideal gas is an example of a *polytropic* equation:

$$p\,\rho^{-\kappa} = \text{konstant} \tag{C.10}$$

The quantity κ is the *polytropic exponent*, and for an ideal gas $\kappa = \gamma$. Such equations of state are also used as approximations for real (in the thermodynamical sense!) gases, where adiabatical (isentropic) processes may be described by effective constant values for κ for given intervals in p and T.[3] This type of equation is mentioned in connection with the Bernoulli equation (**??**) applied to a gas.

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[3]Notice that for a real gas, $\kappa \neq \gamma$ [Finjord 1988], not to mention that Eq. (C.10) with a constant κ is not exact for adiabatical processes in real gases!