Appendix D

Polar and axial vectors

These notions refer to the properties of vectors under *reflections*, their *parity properties*. One can consider reflection of the physical system in a plane [Hylleraas 1950], with the coordinate system fixed, or equivalently inversion of all coordinate axes [Goldstein 1980] with the physical system fixed. We will use the latter in what follows.

Let $\boldsymbol{r}$ denote a position vector in a 3D Euclidean space, and $\boldsymbol{p}$ a momentum vector in the point. After inversion of the coordinate directions the two vectors' directions in space will be unchanged, so the transformation rule will be:

- *Polar* vectors:

$$\begin{aligned} x_i &\rightarrow x_i' = -x_i \\ p_i &\rightarrow p_i' = -p_i \end{aligned} \tag{D.1}$$

The angular momentum $\boldsymbol{L}$,

$$\begin{aligned} \boldsymbol{L} &= \boldsymbol{r}\times\boldsymbol{p} \\ L_i &= \epsilon_{ijk} x_j p_k \end{aligned}$$

has a different rule of transformation:

- *Axial* vectors:

$$\begin{aligned} L_i \rightarrow L_i' &= \epsilon_{ijk} x_j' p_k' \\ &= \epsilon_{ijk} x_j p_k \\ &= L_i \end{aligned} \tag{D.2}$$

The components of polar vectors change sign during the inversion, those of axial vectors do not. Among examples of polar vectors in these lecture notes:

$$\boldsymbol{r}\ ,\ \nabla\ ,\ \boldsymbol{u}$$

One example of an axial vector is the vorticity:

$$\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$$

(also a vector product). The vector product of two polar or two axial vectors will be an axial vector; considerations of sign will correspondingly show that the vector product of one polar and one axial vector will be a polar vector.[1]

Axial vectors have an associated direction of rotation, a *handedness*: An axial vector can equivalently be expressed by the components of an antisymmetric tensor. See the example in Appendix B,

$$\eta_{ij} = \frac{1}{2}\epsilon_{ijk}\omega_k$$

where the antisymmetric tensor η_{ij} is a measure for the rotational property of an arbitrary velocity field, like $\boldsymbol{\omega}$.

A lefthanded coordinate system results from the inversion of a righthanded one. An axial vector thus changes handedness because its components, referred to the coordinate systems, do not change.

If one side of a vector equation decribing a physical relation is polar (or axial), then also the other side must be polar (or axial). That follows because a physical process and a reflected copy of it must be physically equivalent.[2] In Chapter 4 that requirement is used in the derivation of the velocity field for Stokes flow.

*

In the preceding text we got a glimpse of some fundamental relations between vectors and the structures called tensors.[3] Both can be related to the notion of *differential forms*, mathematical structures which however are well beyond the scope of these lecture notes. The introduction of tensors in this course is of the traditional operational kind for fluid mechanics: They are structures which arise in a natural way in the description of tensions in a surface, including supporting structures like Levi-Civitá's which contribute to simplifying the notation. On that background it is not easy to give a consise reason for an equation like (B.12), beyond stating that it has a form which does not favorize any direction in space compared to the others. In other derivations of Eq. (B.13) at the same level of description, see for instance [Papatzacos 2003], one neither avoids postulates.

The energetic student is therefore advised to indulge in a course of diffential surface geometry!

[1] Instead of polar and axial vectors, mathematicians use the notions *vectors* and *bivectors*.

[2] Be warned: That isn't the case for *weak interactions* in atomic physics.

[3] The etymological origin of these monsters with 2 or more indices which from time immemorial have terrified students, is precisely in the mechanics of continuous media, where the *tension* in a surface has to be specified by two directions in space—that of the force and that of the surface normal.