Appendix E

Waves

E.1 Elementary plane-wave formalism

Consider the expression[1]

$$A\,e^{i\theta}\ ,\quad \theta = \boldsymbol{k}\boldsymbol{r} - \omega t + \alpha \tag{E.1}$$

where $\exp{(i\theta)} = \cos\theta + i\sin\theta$, $\boldsymbol{r}$ being a position vector and t the time, and α a constant. For a fixed $t = t_0$ the *phase* θ will have a fixed value $\theta = \theta_0$ everywhere in a given plane with $\boldsymbol{k}$ as a normal vector, since $\boldsymbol{r}$'s components parallel to the plane do not contribute to the scalar product. Without any loss of general applicability we will in what follows assume

$$\boldsymbol{k} = k\hat{\boldsymbol{e}}_1$$

with the plane's position at $x = x_0$ for $t = t_0$. For later times $t = t_0 + \Delta t$ we have $\theta = \theta_0$ for $x = x_0 + \Delta x$, and $\exp(i\theta)$ will still have the initial value for all $(\Delta t,\, \Delta x)$ for which

$$kx_0 - \omega t_0 + \alpha + k\Delta x - \omega\Delta t = \theta_0 + 2\pi N\ , \qquad N \text{ integer}$$

Points with a constant phase will thus be moving in the positive x direction with the *phase velocity*

$$V = \frac{dx}{dt} = \frac{\Delta x}{\Delta t} = \frac{\omega}{k} \tag{E.2}$$

For a constant $t = t_0$, $\exp(i\theta)$ will have the same value for all

$$x = x_0 + N\lambda\ , \qquad N \text{ integer}$$

and for a constant $x = x_0$, $\exp(i\theta)$ will have the same value for all

$$t = t_0 + NT\ , \qquad N \text{ integer}$$

with λ and T the *wavelength* and the *period*, respectively. Further definitions follow:[2]

$$k = \frac{2\pi}{\lambda} \qquad \textit{wave number} \tag{E.3}$$

$$\omega = \frac{2\pi}{T} = 2\pi f \qquad \textit{angular frequency} \tag{E.4}$$

[1] This summary is included for the benefit of eventual students without physics as a specialty.
[2] The angular frequency symbol not to be mistaken for the vorticity!

Thus, Eq. (E.1) describes a periodic field of values moving in the $\boldsymbol{k}$ direction with velocity V, a *wave*. Since only one frequency is involved, this wave is called *monochromatic*, and a *plane wave* since all points in a given plane normal to $\boldsymbol{k}$ have the same value for the expression for a given t. If the *amplitude* A is replaced by a vector $\boldsymbol{A}$, the wave is *transversal* for $\boldsymbol{A} \perp \boldsymbol{k}$ and *longitudinal* for $\boldsymbol{A} \parallel \boldsymbol{k}$.[3]

The physical quantities of the wave are implicitly represented by either the real or the imaginary part of (E.1). The real part has been used explicitly in Chapters 5 and 6 to describe the wave, in the case of the linear surface waves with the amplitude a function of the depth:

$$A(z)\cos(kx - \omega t)$$

The wave number is $2\pi \times$ the number of wave crests per unit length. If the medium of propagation of the wave is moving relative to an observer, this person will count a different number of tops passing per unit time compared to the case where the medium is at rest. This change in (angular) frequency is called the *Doppler effect*, and by counting crests one can easily satisfy oneself that the change in frequency is

$$\overline{\omega} - \omega = \boldsymbol{k} \cdot \boldsymbol{u}_s \tag{E.5}$$

where $\boldsymbol{u}_s$ is the medium's transport velocity.

The functional form of the frequency's dependence on the wave number is called the *dispersion relation*:

$$\omega = \omega(k) \tag{E.6}$$

For a constant phase velocity, $\omega(k) = Vk$.

E.2 Group velocity

For a wave with the dispersion relation $\omega = \omega(k)$ the *group velocity* is defined by

$$U = \frac{d\omega}{dk} \tag{E.7}$$

Its relation to the phase velocity V is

$$\begin{aligned} \omega &= Vk \\ d\omega &= V dk + k\frac{dV}{dk}\, dk \\ U &= V + k\frac{dV}{dk} \end{aligned}$$

Or:

$$U = V - \lambda\frac{dV}{d\lambda} \tag{E.8}$$

For a wave composed by plane waves with different frequencies, evidently the group velocity will contain more information than the phase velocity of a wave of any single wave. Let us consider the physical interpretation of the group velocity:

[3] Surface waves in a liquid is an example of an intermediate case, concerning the movement of the fluid particles.

We start by a wave given by a *Fourier superposition* (see also Appendix G) of plane waves with frequencies in a narrow interval of width 2ϵ around a central frequency ω_0.[4] With the complex formalism and the real part chosen as representing the physically relevant quantity, we denote the wave's deviation from the equilibrium position by

$$\zeta(x,t) = \int_{k_0-\epsilon}^{k_0+\epsilon} a(k) e^{i(kx-\omega t)}\, dk \tag{E.9}$$

The exponent can be rewritten as

$$\begin{aligned} kx - \omega t &= k_0 x - \omega_0 t + (k-k_0)x - (\omega - \omega_0)t \\ &= (k_0 x - \omega_0 t) + (\Delta k\, x - \Delta\omega\, t) \end{aligned}$$

Thus,

$$\zeta(x,t) = C(x,t) e^{i(k_0 x - \omega_0 t)} \qquad , \qquad C(x,t) = \int_{k_0-\epsilon}^{k_0+\epsilon} a(k) e^{i(\Delta k\, x - \Delta\omega\, t)}\, dk \tag{E.10}$$

This can be interpreted as a high "carrier frequency" ω_0 modulated by a time dependent amplitude $C(x,t)$. The propagation velocity of this amplitude is

$$\begin{aligned} \frac{dx}{dt} &= \frac{x}{t} = \frac{\Delta\omega}{\Delta k} \\ &\approx \frac{d\omega}{dk} \end{aligned} \tag{E.11}$$

which is precisely the group velocity U. Depending on the sign of $dV/d\lambda$, the group velocity may be smaller or larger than the phase velocity.

The modulated shape of the wave can be interpreted as a representation of *information*, or of *energy* if the latter quantity is related to the deviation from the equilibrium position. We may therefore interpret U as the propagation velocity of the energy and information of a wave. For a visualization, let the Fourier superposition be a pure addition of two simple adjacent frequencies, with $k_1 \approx k_2 \approx k$ and $\omega_1 \approx \omega_2 \approx \omega$:

$$\begin{aligned} &\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t) \\ &= 2\cos\frac{(k_1+k_2)x - (\omega_1+\omega_2)t}{2} \cos\frac{(k_1-k_2)x - (\omega_1-\omega_2)t}{2} \\ &\approx 2\cos(kx - \omega t)\cos(\frac{1}{2}\Delta k\, x - \frac{1}{2}\Delta\omega\, t) \end{aligned} \tag{E.12}$$

This is a rapidly varying signal with frequency ω, with a superposed slowly varying signal with frequency $\frac{1}{2}\Delta\omega$. The latter will have *nodes* where the deviation from the equilibrium position is zero. If the wave's energy is related to the deviation, then the energy cannot pass through a node of the wave with frequency $\frac{1}{2}\Delta\omega$.

Thus, the energy propagates together with this superposed wave whose group velocity is $\Delta\omega/\Delta k$. If ω is not proportional to k, the group velocity will differ from the phase velocity ω/k. For surface waves on water this was shown explicitly for the case of a single plane wave. (The same holds in the limit $\Delta k, \Delta\omega \to 0$.)

If a superposition is to represent an "isolated" wave, a *wave packet*, one cannot stick to a narrow range of frequencies. In this case the integration has to run over all wave numbers from $-\infty$ til $+\infty$.

[4]The final reault will hold equally well for a single plane wave with a nontrivial dispersion relation, with plane surface waves in a liquid (Chapter 5) as an example.