Appendix F

Surface tension and capillary pressure

Suppose that the equilibrium position of a liquid's surface or boundary towards another fluid is located in the xy plane, as visualized in Figure F.1. It has a *surface tension* α, whose dimensionality is force per unit length. The pressures above ($z > 0$) and below the surface are p_0 and p, respectively, with p_0 often assumed to be a constant. For a surface deflection $\delta\zeta = \delta\zeta(x, y)$, considered positive in the direction of the fluid with pressure p_0 (i.e. upwards), a change δf of the surface's area will result.

The work δR connected with an infinitesimal surface deformation is the sum of a translational contribution and a contribution due to stretching. At equilibrium, assuming no external energy supplied, this work is zero:

$$\delta R = \int (p_0 - p)\, \delta\zeta \, df + \alpha \delta f = 0 \tag{F.1}$$

The variation of the surface area, δf, can be expressed by the variation $\delta\zeta$ of the surface position. By partial integration, and assuming that at the boundary the surface doesn't move ($\delta\zeta = 0$):

$$\begin{aligned} f &= \int \sqrt{1 + (\frac{\partial\zeta}{\partial x})^2 + (\frac{\partial\zeta}{\partial y})^2}\, dx\, dy \\ &\approx \int \{1 + \frac{1}{2}(\frac{\partial\zeta}{\partial x})^2 + \frac{1}{2}(\frac{\partial\zeta}{\partial y})^2\}\, dx\, dy \end{aligned}$$

Figure F.1: Definitions related to surface tension

$$\begin{aligned}\delta f &\approx \int \{\frac{\partial\zeta}{\partial x}\frac{\partial\delta\zeta}{\partial x} + \frac{\partial\zeta}{\partial y}\frac{\partial\delta\zeta}{\partial y}\}\, dx\, dy \\ &= -\int (\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2})\,\delta\zeta\, dx\, dy \end{aligned} \tag{F.2}$$

By insertion of (F.2) into (F.1) the condition on the integral can be transformed into one on the integrand, since the displacement $\delta\zeta$ can be arbitrary function of the horizontal position:

$$\delta R = -\int \{p - p_0 + \alpha(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2})\}\,\delta\zeta\, dx\, dy = 0$$

eller

$$p - p_0 = -\alpha(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}) \tag{F.3}$$

The difference $p - p_0$ is the *capillary pressure*. As shown in [Landau and Lifshitz 1987][1] this equation corresponds to the *Laplace formula*

$$p - p_0 = \alpha(\frac{1}{R_1} + \frac{1}{R_2}) \tag{F.4}$$

where the principal radii of curvature R_1 vs. R_2 are positive when pointing towards the fluid with pressure p, i.e., in the negative z direction.

[1]The notation in that book is untimely. The indices 1 and 2 are being used to denote the principal radii of curvature (in two orthogonal directions in the xy plane), but also for the two fluids (positions in the z direction).