Appendix G

Multidimensjonal Fourier integrals

Let $\boldsymbol{r} = (x_1, x_2, \ldots, x_n)$ be a vector in an n-dimensional Euclidean space. Under certain conditions [Stephenson 1970] on the existence of the integrals, a function $f(\boldsymbol{r})$ of the components of the vector can be expressed as a *Fourier integral* over another function $f(\boldsymbol{k})$ of a vector $\boldsymbol{k} = (k_1, k_2, \ldots, k_n)$, and vice versa:[1]

$$f(\boldsymbol{r}) = \frac{1}{(2\pi)^n} \int f(\boldsymbol{k}) e^{i\boldsymbol{kr}} \, d^n k \tag{G.1}$$

$$f(\boldsymbol{k}) = \int f(\boldsymbol{r}) e^{-i\boldsymbol{kr}} \, d^n r \tag{G.2}$$

For transformation of functions of time to functions of angular frequency we will use the opposite sign convention:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} \, d\omega \tag{G.3}$$

$$f(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt \tag{G.4}$$

It has been shown for $n = 1$ (without complex notation, though) in [Stephenson 1970] how such integral representations can be derived from the wellknown formalism where $f(x)$ is a discrete sum and $f(k)$ an integral over a finite interval. We will interpret $\boldsymbol{k}$ as a *wave number vector*, since it gives consistency if one makes such a spatial Fourier decomposition and wishes to consider the factor $\exp(i\boldsymbol{kr})$ to be the spatially varying factor in a plane wave[2] (see Appendix E). It is not uncommon (although sloppy from a mathematical point of view) to use the same symbol for the function and its transform, as is done here, even if they will in general have wholly different funtional forms. At the same time one must of course specify the argument to make the distinction between $\boldsymbol{r}$ and $\boldsymbol{k}$ space, or between t and ω space.

By insertion of $f(\boldsymbol{r})$ in the integral for $f(\boldsymbol{k})$ we obtain the Fourier representation of *Diracs's delta function* $\delta(x)$ by requiring consistency:

$$\delta^n(\boldsymbol{r}) = \delta(x_1)\delta(x_2)\ldots\delta(x_n) \tag{G.5}$$

[1] Confusion of conventions abounds. In many texts the minus sign in the exponent has been interchanged between the integrals, and/or the factor $1/(2\pi)^n$ has been moved to the other integral. That is inessential if one keeps consistently to one convention. In [Tritton 1988] the $1/2\pi$ factor has disappeared completely (*that* is impermissible). The convention in the present lecture notes corresponds to that of [Landau and Lifshitz 1987].

[2] Opposite sign conventions for simultaneous transformations in space and time then gives an $\exp i(\boldsymbol{kr} - \omega t)$ factor, a plane wave propagating in the *positive* $\boldsymbol{k}$ direction.

$$\delta(x) \quad = \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, dx \tag{G.6}$$

The delta function has the properties [Rottmann 1984]

$$\delta(x) \;=\; \begin{cases} \infty & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{cases} \tag{G.7}$$

$$\int_{-\infty}^{\infty} \tilde{f}(x)\delta(x-y) \, dx \;=\; \tilde{f}(y) \tag{G.8}$$

The exponential factor in the integrand is the sum of a real and even (symmetric) function and an imaginary and odd (antisymmetric) function:

$$e^{i\omega t} = \cos \omega t + i \sin \omega t$$

It then follows from (G.4) that:

- For $f(t)$ even ($f(t) = f(-t)$) and real, $f(\omega)$ becomes even and real
- For $f(t)$ odd ($f(t) = -f(-t)$) and real, $f(\omega)$ becomes odd and imaginary