Appendix H

Department for tricks

One will run into the same type of calculations and expressions in quite varied environments in mathematical physics. Often it is standard expressions which it may pay off to remember. An assortment from the menu in these lecture notes, of varying degrees of triviality, has been compiled in the following.

Cartesian derivatives of powers of the radius:

Let $r = |\boldsymbol{r}| = (x_j x_j)^{\frac{1}{2}}$. Then

$$\begin{aligned} \partial_i(r^N) &= \frac{N}{2}(x_j x_j)^{\frac{N}{2}-1}\partial_i(x_j x_j) \\ &= N r^{N-2}\delta_{ij}x_j \\ &= N r^{N-2}x_i \qquad\qquad (N \neq 0) \end{aligned} \tag{H.1}$$

The special case for $N = 1$, $\partial_i r = x_i/r$, occurs most often. Furthermore:

$$\begin{aligned} \partial_j\partial_i(r^N) &= N\left\{r^{N-2}\partial_j x_i + x_i\partial_j r^{N-2}\right\} \\ &= N\left\{\delta_{ij}r^{N-2} + (N-2)r^{N-4}x_i x_j\right\} \qquad (N \neq 0) \end{aligned} \tag{H.2}$$

Introducing $\hat{\boldsymbol{n}} = \boldsymbol{r}/|\boldsymbol{r}|$ one gets:

$$\begin{aligned} \partial_i \hat{n}_k &= \frac{1}{r}\partial_i x_k + x_k\partial_i\frac{1}{r} \\ &= \frac{1}{r}(\delta_{ik} - \hat{n}_i\hat{n}_k) \end{aligned} \tag{H.3}$$

Transformation to a total divergence:

In Chapters 2 and 11 derivations occur where an integrand or another expression is reduced to total divergences. That may result in a considerable simplification if for instance Gauss's theorem may be used. As an example, for $\boldsymbol{a}$ a vector and b a scalar one has

$$\boldsymbol{a}\cdot\mathrm{grad}\, b = \mathrm{div}\,(b\boldsymbol{a}) - b\,\mathrm{div}\,\boldsymbol{a} \tag{H.4}$$

A total divergence results on the RHS if $\mathrm{div}\,\boldsymbol{a} = 0$. That is the case for the velocity field in an incompressible fluid, but also for magnetic field strength, and for electric field strength in a region of space with no charges.

Averaging over an isotropically fluctuating vector in correlation functions:

Consider an average over the product of a fluctuating scalar in one point and an isotropically fluctuating vector in another point. The correlation function will be vectorial, but after the averaging there can be no dependence on the direction of the averaged vector. The only remaining information about directions rests in the position difference vector $\boldsymbol{r}$ between the two points. When the scalar and the vector represents pressure and velocity, respectively, the expression used to deduce the correlation equation in Chapter 13 results:

$$\overline{p_1\boldsymbol{u}_2} = f(r)\boldsymbol{r}/|\,\boldsymbol{r}\,| \tag{H.5}$$

Derivation of correlation functions:

Define $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$. Assuming homogeneity, the following relation holds for double correlation functions, with the notation of Chapter 13:

$$\frac{\partial}{\partial x_i}\overline{u_{1j}u_{2k}} = -\overline{\frac{\partial u_{1j}}{\partial x_{1i}}u_{2k}} = \overline{u_{1j}\frac{\partial u_{2k}}{\partial x_{2i}}} \tag{H.6}$$

That is because the velocity in point 1 is independent of the position of point 2, and vice versa.

The surface area of the unit sphere in n dimensions:

When expressing Cartesian coordinates in an n-dimensional space by polar coordinates [Magnus and Oberhettinger 1949]

$$\begin{aligned} k_1 &= k\cos\theta_1 \\ k_2 &= k\sin\theta_1\cos\theta_2 \\ &\vdots \qquad\qquad (0\le\theta_i\le\pi,\ 0\le\phi\le 2\pi,\ 0\le k\le\infty) \\ k_{n-2} &= k\sin\theta_1\ldots\sin\theta_{n-3}\cos\theta_{n-2} \\ k_{n-1} &= k\sin\theta_1\ldots\sin\theta_{n-3}\sin\theta_{n-2}\cos\phi \\ k_n &= k\sin\theta_1\ldots\sin\theta_{n-3}\sin\theta_{n-2}\sin\phi \end{aligned} \tag{H.7}$$

the volume element in a Fourier integral can be written as

$$d^n k = k^{n-1}\,dk\,d\Omega_n \tag{H.8}$$

$$d\Omega_n = \sin^{n-2}\theta_1\sin^{n-3}\theta_2\ldots\sin\theta_{n-2}\,d\theta_1\,d\theta_2\ldots d\theta_{n-2}\,d\phi \tag{H.9}$$

where $k = |\,\boldsymbol{k}\,|$. The integral of $d\Omega_n$ gives the surface area of the unit sphere. It can be expressed by the *gamma function* Γ [Rottmann 1995], and the familiar values for Ω_2 and Ω_3 result:

$$\Omega_n = \int d\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{H.10}$$

$$\Gamma(1) = 1 \quad , \quad \Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi} \tag{H.11}$$

Fourier transform of a spherically symmetrical function:

Consider Eq. (G.1) with the $\boldsymbol{k}$ space function dependent only on $k = |\boldsymbol{k}|$:

$$f(\boldsymbol{r}) = \frac{1}{(2\pi)^n} \int f(k) e^{i\boldsymbol{kr}} \, d^n k \tag{H.12}$$

Introduce polar coordinates in this n-dimensional $\boldsymbol{k}$ space, as in the preceding section, with the k_1 axis parallel to $\boldsymbol{r}$. The integral over all angles except θ_1 gives (see (H.10) and [Rottmann 1995]):

$$\frac{\Omega_n}{\int_0^\pi \sin^{n-2}\theta \, d\theta} = \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}$$

The θ_1 integral gives, once more with some help from Rottmann, a result containing a *Bessel function*:[1]

$$\int_0^\pi e^{ikr\cos\theta} \sin^{n-2}\theta \, d\theta = \frac{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}{(kr/2)^{(n-2)/2}} \, J_{(n-2)/2}(kr)$$

The Fourier transformation (H.12) has thus been reduced to an 1-dimensional integral over k:

$$f(\boldsymbol{r}) = \frac{2^{1-n}}{\pi^{n/2}} \int_0^\infty k^2 f(k) \, \frac{J_{(n-2)/2}(kr)}{(kr/2)^{(n-2)/2}} \, dk \tag{H.13}$$

We notice that $f(\boldsymbol{r})$ will in reality depend only on $|\boldsymbol{r}|$: The Fourier transform of a spherically symmetrical function will also be a spherically symmetrical function.[2]

The small-distance limit of the Bessel function $J_\nu(x)$ is [Rottmann 1995]

$$\lim_{r\to 0} \frac{J_\nu(x)}{(x/2)^\nu} = \frac{1}{\Gamma(\nu+1)}$$

so that[3]

$$f(0) = \frac{2^{1-n}}{\pi^{n/2}\Gamma(\frac{n}{2})} \int_0^\infty k^2 f(k) \, dk \tag{H.14}$$

The special cases for $n = 3$ are:

$$f(\boldsymbol{r}) = \frac{1}{4\pi^{3/2}} \int_0^\infty k^2 f(k) \, \frac{J_{1/2}(kr)}{(kr/2)^{1/2}} \, dk \tag{H.15}$$

$$f(0) = \frac{1}{2\pi^2} \int_0^\infty k^2 f(k) \, dk \tag{H.16}$$

[1] The Bessel function $J_\nu(x)$ satisfies the differential equation for *cylinder functions* $Z_\nu(x)$:

$$\frac{d^2 Z_\nu}{dx^2} + \frac{1}{x}\frac{dZ_\nu}{dx} + (1 - \frac{\nu^2}{x^2}) Z_\nu = 0$$

See [Abramowitz and Stegun 1965] concerning Bessel functions and other special functions.

[2] Which can also be inferred directly from (H.12) by symmetry considerations: After the integration over $\boldsymbol{k}$'s direction, $\boldsymbol{r}$ can only occur in scalar products with itself, if $f(k)$ is a scalar function.

[3] This result follows also directly from (H.12), by choosing $\boldsymbol{r} = 0$ and using (H.10).