Chapter 8

Laminar boundary layers

As treated in a previous chapter, at large $\mathcal{R}$ a flow can be considered nonviscous (and often irrotational), except in a *boundary layer* with thickness δ at the flow boundaries. In this layer the gradient of the tangential velocity cannot be neglected:

$$\frac{\delta}{L} \sim \mathcal{R}^{-1/2}$$

Vorticity is created at the boundaries because of the no-slip condition (**??**). The boundary layer can be considered as the region where the vorticity, which spreads out into the fluid by diffusion due to the term $\nu\nabla^2\boldsymbol{\omega}$ in the vorticity equation (**??**), is of considerable magnitude.

In this chapter we will present a mathematical description of *laminar* boundary layers.[1] In a later chapter we will see that laminar boundary layers may develop into *turbulent* boundary layers because of instabilities, but here we will consider the situation *before* the transition takes place.

8.1 The boundary layer approximation

We will first obtain a simpler and more easily solvable (but still nonlinear) version of the equations of motion in the boundary layer, based on arguments about orders of magnitude. We will make the following simplifications and assumptions:

- Stationary 2D flow
- The x coordinate along a flat wall, with origo at the start of the wall, and y perpendicular to the wall (see Figure 8.1)
- A known free-flow solution outside of the boundary layer ($u_0(x)$ and $p_0(x)$ solutions of the Euler equation)

Continuity equation, with boundary layer scales inserted:

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \qquad (8.1) \\ \frac{U}{L} \qquad \frac{V}{\delta} & \end{aligned}$$

[1] These standard topics are covered by for instance [Landau and Lifshitz 1987], [Tritton 1988] and [Acheson 1990]. In these lecture notes we put particular emphasis on arguments about scaling, which is a basic concept in physics.

Figure 8.1: Boundary layer on a flat channel wall, with scales for length, velocity and pressure differences indicated

Or:

$$V \sim \frac{\delta}{L} U \tag{8.2}$$

- In the boundary layer the velocity component normal to the wall is negligible compared to the component parallel to the wall

The x component of the Navier-Stokes equation, also with scales in the boundary layer indicated:

$$\begin{array}{ccccc} u\dfrac{\partial u}{\partial x} & + \; v\dfrac{\partial u}{\partial y} & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\dfrac{\partial^2 u}{\partial x^2} + \nu\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{U^2}{L} & \dfrac{VU}{\delta} \sim \dfrac{U^2}{L} & & \dfrac{\Pi}{\rho L} \quad \nu\dfrac{U}{L^2} \quad \nu\dfrac{U}{\delta^2} \end{array} \tag{8.3}$$

Likewise, for the y component:

$$\begin{array}{ccccccccc} u\dfrac{\partial v}{\partial x} & + & v\dfrac{\partial v}{\partial y} & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} & + & \nu\dfrac{\partial^2 v}{\partial x^2} & + & \nu\dfrac{\partial^2 v}{\partial y^2} \\ \dfrac{UV}{L} \sim \dfrac{U^2\delta}{L^2} & & \dfrac{V^2}{\delta} \sim \dfrac{U^2\delta}{L^2} & & \dfrac{\Lambda}{\rho\delta} & & \nu\dfrac{V}{L^2} \sim \nu\dfrac{U\delta}{L^3} & & \nu\dfrac{V}{\delta^2} \sim \nu\dfrac{U}{L\delta} \end{array} \tag{8.4}$$

In both Eqs. (8.3) and (8.4) the next to last term on the RHS will be negligible compared to the last term. In both equations the pressure term will have the same order of magnitude as the largest of the other terms:

$$\frac{\Pi}{\rho L} \sim \frac{U^2}{L} \sim \nu\frac{U}{\delta^2}$$
$$\frac{\Lambda}{\rho\delta} \sim \frac{U^2\delta}{L^2} \sim \nu\frac{U}{L\delta}$$

Or:

$$\frac{\Lambda}{\Pi} \sim (\frac{\delta}{L})^2 \tag{8.5}$$

- The pressure difference across the boundary layer is negligible compared to the pressure difference over a length L along the wall

For all y in the boundary layer we may therefore rewrite Eq. (8.3) as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p_0}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{8.6}$$

where u_0 and p_0 outside the boundary layer, which do not depend on y, are related via the Euler equation:

$$u_0\frac{\partial u_0}{\partial x} = -\frac{1}{\rho}\frac{\partial p_0}{\partial x} \tag{8.7}$$

The x component of the Navier-Stokes equation has thus been reduced, since it contains only u and v as unknowns. We insert Eq. (8.7) into it, and together with the contionuity equation we then have two equations for the two unknowns, the *boundary layer equations*:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_0\frac{\partial u_0}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{8.8}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{8.9}$$

8.1.1 Scale invariance

Let us introduce dimensional variables by introducing scales, with $\mathcal{R} = \hat{u}_0 L/\nu$, where $\hat{u}_0$ is a constant velocity which may for instance be the velocity of the incoming fluid at infinite distance:

$$x = Lx' \qquad y = \frac{Ly'}{\sqrt{\mathcal{R}}} \qquad u = \hat{u}_0 u' \qquad v = \frac{\hat{u}_0 v'}{\sqrt{\mathcal{R}}} \qquad u_0 = \hat{u}_0 u'_0 \tag{8.10}$$

Insertion into the boundary layer equations gives

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = u'_0\frac{\partial u'_0}{\partial x'} + \frac{\partial^2 u'}{\partial y'^2} \tag{8.11}$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \tag{8.12}$$

The boundary conditions become:

$$u' = v' = 0 \qquad (y = 0\ ,\quad x \geq 0) \tag{8.13}$$

$$\lim_{y\to\infty} u' = u'_0(Lx') \tag{8.14}$$

These expressions are independent of the viscosity, accordingly their solution cannot depend on the Reynolds number which depends on the viscosity. By a change of $\mathcal{R}$ via ν the new flow pattern in the boundary layer will be given by a *similarity transformation*: The real (unmarked) distances and velocities parallel to the wall will be unchanged, while the real distances and velocities normal to the wall will change proportional to $\sqrt{\mathcal{R}}$.

A particular simplification occurs when $u_0(x)$ has such a form that the RHS of Eq. (8.14) and also $u_0 \partial u_0/\partial x$ are independent of L. That is the case for instance for a semi-infinite plate, where there is no characteristic length scale L. Accordingly, the solutions for u' and v' must have a *scale invariant* form,[2] which depends on x' and y' in such a way that L cancels out. According to Eq. (8.10) this can only be the case for the combination

$$\frac{y'}{\sqrt{x'}} = \sqrt{\frac{\mathcal{R}}{L}} \frac{y}{\sqrt{x}} = \sqrt{\frac{\hat{u}_0}{\nu}} \frac{y}{\sqrt{x}} \tag{8.15}$$

Also u and $\sqrt{L}v$ have to be scale invariant, and can only depend on $y/\sqrt{x}$, eventually multiplied with an arbitrary constant. We will use this insight when solving the equations for a special case below, to demonstrate scale invariance.

This is an example of use of a method with a general applicability in physics. Scale invariance turns out to be present in a plentitude of physical phenomena. Observation of or assumption about scale invariance (exact or as an approximation) is an indispensable aid for instance in statistical physics.[3]

8.1.2 Classification of boundary layers

Prior to solving the boundary layer equations, one solves the Euler equation for the corresponding nonviscous problem to get $u_0(x)$, which enters into the first term on the RHS of Eq. (8.8) as well as in a boundary condition. The solution in the boundary layer has a character depending on the functional form of u_0. Briefly mentioned, the following classification is used:

- $\dfrac{\partial p_0}{\partial x} < 0$, $\dfrac{\partial u_0}{\partial x} > 0$: 'Favorable' pressure gradient, accelerating flow
- $\dfrac{\partial p_0}{\partial x} > 0$, $\dfrac{\partial u_0}{\partial x} < 0$: 'Unfavorable' pressure gradient, retarding flow

With a favorable pressure gradient the boundary layers will be thin (their thickness may even decrease downstream). With an unfavorable gradient that is not the case; the boundary layer may then also be exposed to *flow separation*, where streamlines in the boundary layer upstream in infinitesimal distance from the wall will journey out into the fluid at the *separation point* on the wall. If so, the solution of the Euler equation will be invalid, to the extent that it presupposed the flow to follow the wall. In an actual flow problem, both types may be present simultaneously, in different regions of the flow.

We will have to postpone the reason for the abovementioned classification to the next chapter. Just now we will content ourselves with finding an explicit solution of the boundary layer equations for the case with a zero pressure gradient, where the flow phenomena are not qualitatively different from what we may encounter for a favorable pressure gradient.

[2] Loosely speaking, the solution looks in a sense similar irrespectively of the length scale it is studied on.

[3] In the last chapter of these lecture notes we will consider a simple but epoch-making application of this kind.

8.2 The Blasius profile

Let us specialize to a free-flow solution where the first term on the RHS of (8.8) disappears, since the solution implies $\Lambda = \Pi = 0$:

$$u_0(x) \;=\; \hat{u}_0 \;=\; \text{konstant} \tag{8.16}$$

That is a useful approximation for instance for flow parallel to a flat plate in a uniform flow, with $x = 0$ at the plate's upstream edge. The boundary conditions for the solution of(8.8) and (8.9) become[4]

$$u = v = 0 \qquad (y = 0\;,\;\; x \geq 0) \tag{8.17}$$

$$\lim_{y\to\infty} u = \hat{u}_0 \tag{8.18}$$

In the scaled system (8.11)–(8.14) all dependence on L will now cancel. The system of equations describes a *scale invariant* problem; thus we are essentially considering:

- Flow past a semi-infinite plate

Based on the considerations about scale invariance we already know that x' and y' must enter the equations in a particular combination. The solution for the velocity's x component must have the form

$$u \;=\; \hat{u}_0 g(\eta) \quad , \quad \eta \;=\; C\frac{y}{\sqrt{x}} \quad , \quad C \;=\; \sqrt{\frac{\hat{u}_0}{\nu}} \tag{8.19}$$

where $g(\eta)$ is an unknown function. If we introduce a stream function Ψ, we find immediately by using Eqs. (**??**) that it must be given by

$$\Psi \;=\; \hat{u}_0\frac{\sqrt{x}}{C}f(\eta) \quad , \quad g \;=\; \frac{\partial f}{\partial \eta} \;=\; f' \tag{8.20}$$

where another function $f(\eta)$ has been introduced. By further derivations we find the rest of the expressions entering Eq. (8.8):[5]

$$v \;=\; \hat{u}_0\{-f + C\frac{y}{\sqrt{x}}f'\}\frac{1}{2C\sqrt{x}} \tag{8.21}$$

$$\frac{\partial u}{\partial x} \;=\; -\hat{u}_0 f''\frac{Cy}{2x^{3/2}} \tag{8.22}$$

$$\frac{\partial u}{\partial y} \;=\; \hat{u}_0 f''\frac{C}{\sqrt{x}} \tag{8.23}$$

$$\frac{\partial^2 u}{\partial y^2} \;=\; \hat{u}_0 f'''\frac{C^2}{x} \tag{8.24}$$

These expressions can now be inserted into (8.8). Following cancellations, reduction of a common factor, and use of the definition of C, we end up with the *Blasius equation*, a common

[4] Strictly speaking, the condition at infinity might have been formulated as $y \to \pm\infty$, when there is flow on both sides of the plate. This can be repaired in the final expression by replacing η in $f(\eta)$ by $|\eta|$, so that the result may be interpreted as the solution for the flow on both sides of an infinitely thin plate.

[5] By introducing a stream function we have implicitly presupposed continuity. Eq. (8.9) will then be trivially satisfied (just try!), and we do not need to bother about it any more.

nonlinear differential equation:[6]

$$ff'' + 2f''' = 0 \tag{8.25}$$

The corresponding transformed boundary conditions become

$$f = f' = 0 \qquad (\eta = 0) \tag{8.26}$$

$$\lim_{\eta\to\infty} f' = 1 \tag{8.27}$$

The solution of this nonlinear system can only be obtained numerically.[7] It is known as the *Blasius profile*[8] and is shown by the solid curve in Figure 8.2. It has the properties[9]

$$f'(\eta) = 0.99 \quad \text{for} \quad \eta = 4.99\ldots \tag{8.28}$$

$$\lim_{\eta\to 0} f''(\eta) = 0.332\ldots \tag{8.29}$$

By using the convention about 1 % deviation from the free-flow value as a definition of the boundary layer thickness, we determine the proportionality constant in Eq. (**??**):[10]

$$y \;\to\; \delta = 4.99\sqrt{\frac{\nu}{\hat{u}_0}}\sqrt{x} \tag{8.30}$$

By defining different Reynolds numbers

$$\mathcal{R}_x = \frac{\hat{u}_0 x}{\nu} \tag{8.31}$$

$$\mathcal{R}_\delta = \frac{\hat{u}_0 \delta}{\nu} \tag{8.32}$$

[6] In for example [Bender and Orszag 1978] and [Acheson 1990] the equation used without the factor 2 in front of f''', however with the same boundary conditions. This is no misprint, but rather a result of the equation's similarity properties: If $ff'' + \alpha f''' = 0$, then introduce

$$f(\eta) = \sqrt{\alpha}\, h(\eta/\sqrt{\alpha})$$

and you will obtain $hh'' + h''' = 0$, $f'(\eta) = h'(\eta/\sqrt{\alpha})$.

[7] In some textbooks in engineering fluid mechanics like [Olson 1980], [Massey 1983] and [Gerhart et al. 1992] various simple analytical expressions approximating $f'(\eta)$ for practical use have been listed. However, they are not solutions of the Blasius equation in some approximation, but rather purely ad hoc coincidental expressions without any analytical relation to the solution of the Blasius equation. To maximise confusion: In for instance [Massey 1983], [Landau and Lifshitz 1987], [Daugherty et al. 1989] and [Gerhart et al. 1992], $f'(\eta)$ has been called $f(\eta)$.

[8] Found by H. Blasius i 1908.

[9] [Massey 1983] cites the value 4.91 instead of 4.99. In both cases one might probably round off to 5 without reducing the 'practical' applicability of the expressions.

[10] Another common definition of the boundary layer thickness is the *displacement thickness* δ^*. By the definition

$$\int_0^\infty (\hat{u}_0 - u)\, dy = \hat{u}_0 \delta^*$$

the Blasius solution produces an expression like (8.30), however with another proportionality constant:

$$\delta^* = 1.721\sqrt{\frac{\nu}{\hat{u}_0}}\sqrt{x}$$

Figure 8.2: Comparison of the theoretical Blasius profile with experimental data

for use outside and in the boundary layer, respectively, we obtain

$$\frac{\delta}{x} = 4.99 \frac{1}{\sqrt{\mathcal{R}_x}} \tag{8.33}$$

$$\mathcal{R}_\delta = 4.99 \sqrt{\mathcal{R}_x} \tag{8.34}$$

In Figure 8.2 also experimental data for 3 different values of $\mathcal{R}_x$ have been plotted. The data is seen to fall on top of each other with reasonable accuracy—a confirmation of scale invariance. The agreement with the explicitly scale invariant theoretical Blasius curve is also excellent.

A flat surface was a presupposition for the calculations leading to the Blasius equation. It can be shown, however (although we will not enter into it) that the Navier-Stokes equation in the boundary layer, expressed in curvilinear coordinates parallel to and perpendicular to a *curved* surface, is of the same form as the one we have treated provided $\delta \ll R$, where R is the radius of curvature of the surface. u_0 must then be expressed by the curvilinear coordinate along the surface. Similar results can then be derived for curved surfaces.

Notice that even if we use the term laminar about this type of boundary layer, it is a condition that the Reynolds number is not too small. Boundary layers, by definition with a thickness much smaller than the typical linear scale of the system, appear in the nonviscous limit $\mathcal{R}_x \gg 1$.

The velocity distribution predicted by the Blasius becomes unstable if $\mathcal{R}_x$ is too large. In a later chapter we will describe the transition and the condition in the *turbulent* boundary layer which then emerges. All boundary layers with $\partial p_0/\partial x = 0$ become unstable if their downstream extent is large enough. However, unless the level of disturbances is not too high, there will be a significant range where the Reynolds number is large enough that a boundary layer will develop, but still small enough that it stays laminar.

The boundary layer equations derived above have also valid applications to some related problems. In at least two cases we can find exact analytical solutions, in fact:[11]

- Laminar 2D jet
- Laminar wake behind a thin plate (asymptotical behavior)

A closing comment: The type of mathematical problems arising in the treatment of flows with a boundary layer, has inspired the development of some special approximation methods in applied mathematics. One of them is even called *boundary layer theory* [Bender and Orszag 1978]. In differential equations where the highest derivative is multiplied by a parameter ϵ, regions with a strongly varying solution ('boundary layers'!) may emerge when $\epsilon \to 0$. Boundary layer theory (of the mathematical variety) has its applications in cases where the extent of these regions approach zero in this limit. There is the analogy with the behavior of the Navier-Stokes equation for $\mathcal{R} \to \infty$.[12]

8.3 Problems

Problem 8.1 Consider a 2D flow away from a stagnation point on a flat wall, where the nonviscous velocity at the wall is given by (see Problem 1.2, part a2)

$$u_0(x) = ax \qquad (a > 0)$$

where x is the distance from the stagnation point along the wall.

a) Show that this is a case of a favorable pressure gradient.

b) Show that the boundary layer below this flow, in regions where $\mathcal{R}_x = ax^2/\nu$ er stort, has a constant thickness.

c) Show that if the stream function in the boundary layer can be written as (with b an adjustable constant > 0)

$$\Psi = kaxf(\frac{y}{b\delta}) \qquad , \qquad \lim_{\eta\to\infty} f'(\eta) = 1$$

then one may write

$$k = b\delta = \sqrt{\frac{\nu}{a}}$$

d) Show that the Navier-Stokes equation's x component in the boundary layer reduces to

$$f''' + ff'' - (f')^2 + 1 = 0$$

and write doen the boundary conditions.

[11] Even if the calculations are simple, it would lead beyond the scope of this course to present them. Estimates of magnitude for a laminar boundary layer can be found in Problem 7.2.

[12] See also [Acheson 1990] p. 269.

Problem 8.2 The growth of a boundary layer on a porous wall can be suppressed by sucking a part of the fluid into the wall. Suppose that under suitable circumstances this may give rise to a boundary layer where both the thickness and the velocity profile stays unchanged downstream in the flow, for 2D flow along a flat wall with a zero external pressure gradient.

a) Determine the velocity profile using the equations of motion and continuity, using the boundary conditions

$$u = 0 \,, \quad v = -v_0 < 0 \qquad (y = 0)$$

$$\lim_{y \to \infty} u = u_\infty$$

b) Where, on a wall with a finite extent, would this equation find its application? And how does the solution relate to the case with $v_0 = 0$?

Problem 8.3 Suppose that the Blasius profile can be applied in practice for a thin flat plate with a finite length L (for $\mathcal{R} = \hat{u}_0 L/\nu \gg 1$), i.e., that the corrections near the downstream edge are negligible. Let the plate's width be b, and let the fluid flow on both sides of the plate.

Show that the total friction force on the plate is then given by dimensional analysis by the friction coefficien C_f as

$$F = \frac{1}{2} C_f \rho \hat{u}_0^2 2bL$$

where

$$C_f = \frac{1.328}{\sqrt{\mathcal{R}}}$$

Problem 8.4 For the Blasius profile, calculations show that in the limit $\eta \to \infty$ the following holds, with an exponentially small error:

$$f(\eta) = \eta - \beta \,, \quad \beta = 1.721\ldots$$

a) Show that with the following definition of the *displacement thickness* δ (see also the footnote on p. 96)

$$\hat{u}_0 \delta^* = \int_0^\infty (\hat{u}_0 - u)\, dy$$

one has

$$\delta^* = \beta \sqrt{\frac{\nu x}{\hat{u}_0}}$$

b) Because the flow becomes retarded in the growing boundary layer, continuity implies that the flow velocity gets an induced *outward* component even outside the boundary layer. Show that for the Blasius profile this induced velocity is

$$\lim_{y \to \infty} v = \frac{1}{2} \beta \sqrt{\frac{\nu \hat{u}_0}{x}} = \frac{1}{2} \frac{\beta}{\sqrt{\mathcal{R}_x}} \hat{u}_0$$

Problem 8.5 Eq. (8.30) can be reformulated as follows:

$$\delta = 4.99\,\frac{\nu}{\hat{u}_0}\sqrt{\mathcal{R}_x}$$

Assume that laminar boundary layers become unstable at $\mathcal{R}_x \sim 10^6$ (an order-of magnitude estimate). Find the resulting order of magnitude for the maximum laminar boundary layer thickness at a flat plate, and how far from the edge it will appear, for

a) air ($\nu \sim 1.5\times10^{-5}\,\mathrm{m^2\,s^{-1}}$) for a free-flow velocity $100\,\mathrm{m\,s^{-1}}$

b) crude oil ($\nu \sim 10^{-4}\,\mathrm{m^2\,s^{-1}}$) for a free-flow velocity $1\,\mathrm{m\,s^{-1}}$