Chapter 3

Flow regimes and equations of motion

In this chapter we will see what form the Navier-Stokes equation has for two limiting flow conditions. In particular we will clarify the reason why the Euler equation may be a good approximation, and why this presupposes the existence of *boundary layers*. We will largely specialize to the case of a stationary flow.

3.1 The Navier-Stokes equation on dimensionless form

Let us scale distance and velocity in the Navier-Stokes equation (**??**) by constant values L and U typical for the flow problem studied, such that x_i' and u_i' are dimensionless and with 1 as characteristic order of magnitude:

$$x_i' = \frac{x_i}{L} \tag{3.1}$$

$$\nabla' = \hat{\boldsymbol{e}}_i \frac{\partial}{\partial x_i'} \tag{3.2}$$

$$u_i' = \frac{u_i}{U} \tag{3.3}$$

A natural scaling of the time variable will then be

$$t' = t\frac{U}{L} \tag{3.4}$$

If in addition we introduce the scalings

$$(\Delta p)' = \frac{\Delta p}{\rho U^2} \tag{3.5}$$

$$\hat{\boldsymbol{g}} = \frac{\boldsymbol{g}}{|\boldsymbol{g}|} \tag{3.6}$$

it is wellknown (or a useful exercise to show) that we then get

$$\frac{\partial \boldsymbol{u}'}{\partial t'} + (\boldsymbol{u}' \cdot \nabla')\boldsymbol{u}' = -\nabla'(\Delta p)' + \frac{1}{\mathcal{F}^2}\hat{\boldsymbol{g}} + \frac{1}{\mathcal{R}}\nabla'^2\boldsymbol{u}' \tag{3.7}$$

with the Reynolds and Froude numbers defined as in Eqs. (**??**) and (**??**). Eq. (3.7) may give the most intuitive reason for the concept of *dynamical similarity*, which is the basis for the use of models in fluid dynamics.

Of particular importance for us in this course is that for example the Reynolds number denotes the ratio between the contributions of two dynamical quantities in the flow, namely, the inertial force and the viscous force:

$$\frac{|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}|}{|\nu\nabla^2\boldsymbol{u}|} \sim \frac{U^2/L}{\nu U/L^2} = \frac{UL}{\nu} = \mathcal{R} \tag{3.8}$$

In the following we will neglect the gravitational force for simplicity, and consider the limiting cases $\mathcal{R} \ll 1$ and $\mathcal{R} \gg 1$. We will see that there is a very important exception from the rule that the scaled and dimensionless quantities in Eq. (3.7) are typically of order of magnitude 1.

3.2 Low Reynolds numbers: Creeping flow

If $\mathcal{R} \ll 1$, the viscous term will dominate so that the inertial term will play an unimportant role. For a stationary flow,

$$0 = -\nabla'(\Delta p)' + \frac{1}{\mathcal{R}}\nabla'^2\boldsymbol{u}' \tag{3.9}$$

eller

$$\nabla p = \mu\nabla^2\boldsymbol{u} \qquad (\mathcal{R} \ll 1) \tag{3.10}$$

The pressure term must be kept since we need 4 unknowns to formulate 4 independent equations (the continuity equation being the fourth). Physically this means that the magnitude of the pressure term is determined by the other dynamically important quantities (in this case, the viscous term).

Eq. (3.10) describes *creeping flow*. It is evidently a considerable simplification, compared to the general nonlinear version of the Navier-Stokes equation. Many exact solutions of (3.10) are known. In the next chapter we will use it to calculate the drag force on a sphere in a flowing fluid. Two characteristic traits of a flow where the equation holds, may be usefully mentioned here:

- *Reversibility*: We notice that if $\boldsymbol{u}(\boldsymbol{r},t)$ with $p = p_0 + \Delta p(\boldsymbol{r},t)$ is a solution, then also $-\boldsymbol{u}(\boldsymbol{r},t)$ with $p = p_0 - \Delta p(\boldsymbol{r},t)$ is a solution. The streamline pictures will be identical, except with the velocity vector arrows and the pressure gradients reversed. A consequence: For a flow with an upstream/downstream mirror symmetry of the constraints, the whole streamline picture will have such a mirror symmetry; however the pressure distribution will be antisymmetric.

- Viscous interactions are *long-range*: As an example, Figure 3.1 shows a velocity profile at the centre plane in a flow perpendicular to a cylinder at a low Reynolds number, with asymptotical velocity $\boldsymbol{u}_0$ far from the cylinder. As will be evident in the next section, there is a considerable between this type of flow and the streamline picture for $\mathcal{R} \gg 1$; see also a problem in the next chapter.

Figure 3.1: Centre plane velocity profiler for a flow with $\mathcal{R} = 0.1$ past a circular cylinder

3.3 High Reynolds numbers: Nonviscous flow

For $\mathcal{R} \gg 1$, Eq. (3.7) reduces to the *Euler equation*:

$$\frac{\partial \boldsymbol{u}'}{\partial t'} + (\boldsymbol{u}' \cdot \nabla')\boldsymbol{u}' = -\nabla'(\Delta p)' \tag{3.11}$$

or

$$\rho D_t \boldsymbol{u} = -\nabla p \qquad (\mathcal{R} \gg 1) \tag{3.12}$$

In this case the "simplification" is mathematically nontrivial compared to what we saw at low Reynolds numbers. The equation is still nonlinear. The neglected term is of the highest order in the derivations; the *order* of the differential equation has been reduced. Then, simultaneously one of the boundary conditions must be relaxed. Since the no-slip condition follows from the viscosity property, we expect that this is the one to let go—obviously in agreement with our knowledge that the impermeability condition (**??**) is traditionally used with the Euler equation.

A paradoxical situation: The Euler equation becomes better as an approximation the larger $\mathcal{R}$ is; however, the no-slip condition (**??**) cannot simply be 'turned off', since the viscous term will always be present even if it is generally unimportant in size, and from experiment we know that the fluid will not slip at the boundaries irrespective of the value of $\mathcal{R}$.

Thus, the viscous term in the Navier-Stokes equation will always be important close to *boundaries* even for $\mathcal{R} \gg 1$, and the region where this holds is called the *grensesjiktet.* There the equation of motion will acquire the order which corresponds to the number of physically motivated boundary conditions. The line of argument in connection with Eq. (3.7) that the viscous force is negligible for $\mathcal{R} \gg 1$, breaks down in the boundary layer. In this region the flow will develop an internal length scale much less than L:

- The *boundary layer thickness* δ

Figure 3.2: Velocity profiles for flow past a thin plate: a) Imaginary nonviscous fluid; b) Real fluid for $\mathcal{R} \gg 1$. Broken lines denote the outer limits of the boundary layers and the wake

For a stationary flow the following holds in the boundary layer (because the flow largely is parallel to the boundary):

$$|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}| \sim \frac{U^2}{L} \tag{3.13}$$

$$|\nu\nabla^2\boldsymbol{u}| \sim \nu\frac{U}{\delta^2} \tag{3.14}$$

We get an estimate for the boundary layer thickness by assuming that the inertial force and the viscous forces are of comparable magnitude:

$$\frac{U^2}{L} \sim \frac{\nu U}{\delta^2} \tag{3.15}$$

or

$$\frac{\delta}{L} \sim (\frac{UL}{\nu})^{-\frac{1}{2}} = \mathcal{R}^{-\frac{1}{2}} \tag{3.16}$$

We will illustrate these condition by two examples of fluid flow restrictions.

Thin plate: Figure 3.2 shows an uniform flow near such a plate. The flow velocity is close to the asymptotical value $\boldsymbol{u}_0$ except near the plate, in a boundary layer on each side of the plate (as well as in a *wake* in the flow behind the plate). There the velocity variation results in large values for $\frac{\partial^2\boldsymbol{u}}{\partial y^2}$ (where y is the distance perpendicular to the plate), so that Eq. (3.14) is a better estimate for the order of magnitude than (3.8) does.

A useful and common *practical* specification of the boundary layer thickness at a plate results from defining δ by

$$y = \delta \qquad \Rightarrow \qquad |\boldsymbol{u}| = 0.99\,|\boldsymbol{u}_0| \tag{3.17}$$

Figure 3.3: Velocity profile at the centre plane for an imaginary nonviscous fluid (broken line) and for a real fluid with $\mathcal{R} \gg 1$ (solid line) in a flow past a cylinder

In this case the natural choice of macroscopical length scale is the distance x which the fluid has flown from the leading edge of the plate. Insertion of $U = u_0$ and $L = x$ into Eq. (3.16) gives

$$\delta \sim \sqrt{\frac{\nu x}{u_0}} \tag{3.18}$$

We will give further reasons for this form in a later chapter by explicite calculation for a *laminar*[1] boundary layer, where also the constant of proportionality is found.

Finally, we notice that the outer limit of the boundary layer is not a streamline. The fluid flow will cross this limit.

Sylinder: Figure 3.3 shows a velocity profile at the midplane in a flow perpendicular to a cylinder, with asymptotical velocity $\boldsymbol{u}_0$ far from the cylinder. sylinderen.[2] It also illustrates that a more general definition of the boundary layer thickness than Eq. (3.17) would be the thickness of the region where the velocity deviates more than 1 % from the pure nonviscous solution: The solution of Euler's equation with the impermeability condition imposed is not necessarily an uniform flow. In this example, Euler's equation predicts the value $2\boldsymbol{u}_0$ for the flow velocity in the centre plane at zero distance from the cylinder (see Problem 3.1).

3.3.1 Validity of Euler's equation and of the potential flow approximation

We have seen that in the limit $\mathcal{R} \gg 1$ the flow outside thin layers at boundaries can be described *without* the no-slip condition, i.e. with the Euler equation and eventually a potential flow description approximately valid beyond a distance δ from the boundary: For $y \sim \delta$ and further away, the influence of viscosity on the flow velocity is so small that no 'no-slip'

[1] Strictly speaking, Eq. (3.18) holds only for laminar boundary layers.

[2] The figure neglects effects due to *boundary layer separation*; see later chapters.

condition holds, except for the fluid in the boundary layer. For large $\mathcal{R}$ the volume of fluid in the boundary layer will be a small part of the total volume of the flowing fluid.

In elementary courses we learn that for large Reynolds numbers the flow becomes unstable, and that turbulence emerges. Here, however, we have asserted that outside of the boundary layer we have laminar flow to a good approximation, even for very large Reynolds numbers. How can this be consistent?[3]

The connection is that instabilities with transition to turbulence appear in the boundary layer, while the outside flow may still be laminar. The 'wellknown' phenomenon of turbulent pipe flow emerges when the boundary layer has grown sufficiently in thickness that it reaches the pipe axis. Other forms of turbulent flow may be attributable to *separation* of the boundary layer, so that it has transported turbulence into the adjacent fluid: Within the boundary layer, viscous effects may eventually lead to *flow separation* from the boundary. If so, it will influence the total flow pattern and evidently also the validity of Euler's equation as an approximation.

Drag effects are due to the viscous forces acting in the boundary layer (as well as eventual separation effects, see later chapters). The Euler equation and potential theory are of course inadequate for calculations of the magnitude of such effects.

3.4 Problems

Problem 3.1 a) Show that the conditions for nonviscous incompressible 2D flow past a circular *cylinder* with radius R, with uniform velocity u_0 perpendicular to the cylinder axis far from the cylinder, is satisfied by the following velocity potential in cylindrical coordinates:

$$\Phi = -u_0\,(r + \frac{R^2}{r})\cos\phi$$

b) Find the pressure distribution at the cylinder's surface.

Problem 3.2 As for Problem 3.1, however for a *sphere* with radius R, with uniform flow velocity in the positive z direction far from the sphere, and with a velocity potential expressed in spherical coordinates by

$$\Phi = -u_0\,(r + \frac{1}{2}\frac{R^3}{r^2})\cos\theta$$

[3] The length scale in $\mathcal{R}$ for pipe flow, like in the boundary layer, is defined perpendicular to the direction of the flow, while for flow along the plate outside of the boundary layer is defined in the direction of the flow. See also Chapter 8.