Chapter 6

Shock analogies

In this chapter we will treat two unrelated wave phenomena, having in common that they may have large amplitudes and are not covered by the usual linear potential theory for surface waves in a liquid. The first is the *hydraulic jump*, which is due to the same fundamental relations as those which give rise to shock waves in gas dynamics—i.e., an introduction to gas dynamics in disguise. The other is *solitary waves* and *solitons*, where nonlinear terms in the wave equation give important contributions. We will observe that some types of solitary waves can be considered to be shock waves in a wider sense.

6.1 The hydraulic jump as a shock wave analogy

Shock waves may occur in supersonic flow of gases. In the rest system of the wave (Figure 6.1)) the gas flow arrives at the standing wave with supersonic velocity, and leaves it at the other

Figure 6.1: Normal shock wave (c = speed of sound)

side with subsonic velocity. The two values of c are different in general, due to different thermodynamic conditions. Across the shock, which has a thickness of a few times the mean free path for the gas molecules, there will be a jump in temperature and pressure, as well as an increase in entropy—a fraction of the mechanical energy has been irreversibly transformed into heat. The shock is *normal* if the gas flow arrives perpendicular to the standing wave, otherwise *oblique*; the latter is usually the case for the shock wave at the nose of a supersonic bullet or plane. The existence of shock waves in a gas follows from conservation of mass and momentum and from the equation of state of the gas.

We will illustrate shock waves in a gas by a quite analogous phenomenon which may occur in liquid flow with a free surface, the *hydraulic jump*—a stationary flow pattern where the liquid's surface rises abruptly over a short distance with turbulent behavior, so that a fast flow with a small fluid height over the bottom is replaced by a slower flow with a larger height. This is indicated in Figure 6.2.[1] For flow in a channel with a horisontal bottom, a hydraulic jump

Figure 6.2: Hydraulic jump in horizontal fluid flow

can occur for an ideal liquid, with constant but different depths on the two sides of the jump. (In practice, the bottom must be slightly sloping for the flow to be stationary.) A hydraulic jump can therefore be considered to be a wave with large wavelength and an amplitude being non-negligible compared to the depth, the way shock waves differ from usual sound waves by a pressure amplitude being non-negligible compared to the equilibrium perssure of the gas.

We will predict the existence of hydraulic jumps by studying a 1D flow of an ideal liquid in a horizontal channel with a constant rectangular cross section and width b. The same relations hold as for shock waves in a gas:

- Conservation of mass
- Conservation of momentum
- A relation between wave speed and liquid depth (corresponding to the compressibility of a gas)

The flow will also satisfy the continuity equation (**??**) and the Euler equation (**??**). In the latter the term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ must be kept, since we include the possibility of large amplitudes. For a flow in the x direction and surface height h over the bottom, with a constant velocity in the cross section, and a constant atmospheric pressure at the surface independent of h so that $p = \rho g(h - z)$, the two equations give:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{6.1}$$

[1] The phenomenon can be observed in a faucet with a flat bottom, where a circular hydraulic jump will often occur at some distance from the point where the water from the faucet tap hits the bottom.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x} \tag{6.2}$$

We introduce

$$\overline{\rho} = \rho h \quad , \quad \overline{p} = \int_0^h p\,dz = \frac{1}{2}\rho g h^2 = \frac{g}{2\rho}\overline{\rho}^2 \tag{6.3}$$

and get two equations which are formally equivalent to those for an adiabatic flow of an ideal gas with $\gamma = 2$ (see Eq. (**??**)):

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial (u\overline{\rho})}{\partial x} = 0 \tag{6.4}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x} \tag{6.5}$$

The similarity between these two equations and the corresponding ones for a gas flow, implies an analogy with the results for a slightly unphysical ideal gas ($\gamma = 2$), where the results for flow without discontinuities can be compared directly:

- Liquid depth in channel flow
  $\Leftrightarrow$
  Gas density in pipe flow with $\gamma = 2$

The momentum flow density is given by Eq. (**??**). The mass and momentum flow rates therefore become:

$$J_m = \rho u b h \tag{6.6}$$

$$J_p = \int_0^h (p + \rho u^2) b\,dz = \frac{1}{2} b\rho g h^2 + b\rho u^2 h \tag{6.7}$$

The continuity equation (expressing conservation of mass) and conservation of momentum between two arbitrarily chosen crosssections normal to the flow, which must hold even of there are discontinuous jumps in liquid height between the cross sections, then imply:

$$u_1 h_1 = u_2 h_2 \tag{6.8}$$

$$u_1^2 h_1 + \frac{1}{2} g h_1^2 = u_2^2 h_2 + \frac{1}{2} g h_2^2 \tag{6.9}$$

Solving these 2 equations with 4 unknowns we may choose for instance (h_1, h_2) or (h_1, u_1) as independenet variables. In the latter case itis useful to introduce the Froude number $\mathcal{F} = u/\sqrt{gh}$:

$$u_1^2 = \frac{1}{2} g \frac{h_2}{h_1}(h_1 + h_2) \tag{6.10}$$

$$u_2^2 = \frac{1}{2} g \frac{h_1}{h_2}(h_1 + h_2) \tag{6.11}$$

$$\frac{h_2}{h_1} = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2\mathcal{F}_1^2} \quad , \quad \mathcal{F}_1 = \frac{u_1}{\sqrt{gh_1}} \tag{6.12}$$

$$\mathcal{F}_2 = \frac{1}{8\mathcal{F}_1^4}\left\{\frac{1}{2} + \sqrt{\frac{1}{4} + 2\mathcal{F}_1^2}\right\} \tag{6.13}$$

Because of the symmetry of the indices in Eqs. (6.8) and (6.9) we would get two equally valid equations by interchanging the indices 1 and 2 in Eqs. (6.12) and (6.13).

We get the mechanical energy flow rate through a cross-section by integrating the energy flow density of Eq. (??), with a term added which gives the potential energy:[2]

$$\begin{aligned} J_E &= \int_0^h \rho u(\frac{1}{2}u^2 + \frac{p}{\rho} + gz) b\, dz \\ &= J_m g\, (h + \frac{1}{2g}u^2) \end{aligned} \tag{6.14}$$

The difference between the energy flow rates in two different cross-sections becomes:

$$\begin{aligned} J_{E2} - J_{E1} &= J_m g\, (h_2 - h_1 + \frac{1}{2g}(u_2^2 - u_1^2)) \\ &= -\frac{1}{4}\, J_m g\, \frac{(h_2 - h_1)^3}{h_1 h_2} \end{aligned} \tag{6.15}$$

Expressed in terms of *head* (energy per unit weight) the energy rate difference becomes:

$$\Delta h_E = -\frac{(h_2 - h_1)^3}{4 h_1 h_2} \tag{6.16}$$

We notice that unless $h_1 = h_2$ (and thus $u_1 = u_2$) there is an energy difference between the two cross-sections, even for an ideal flow and a horizontal bottom! No assumption about the distance between the cross-sections was made, and Eqs. (6.8) and (6.9) include the possibility of an abrupt change (a *jump*). Since no external energy has been added, and a continuous change of height would have been governed by Euler's equation which cannot imply a change of the total mechanical energy, a jump is the only possible case for this idealized flow.[3] Quite analogous to the interpretation in the case of inelastic collisions in elementary point mechanics, conservation of energy in addition to momentum and mass would imply an overdetermined system of equations.

Assume now that the indices 1 and 2 indicate an upstream and a downstream cross-section, respectively. Since an increase in energy is impossible, it follows that $h_2 - h_1 \geq 0$: Downstream of the jump the flow will be slower, with a larger surface height over the bottom.

Of course, for a real liquid the difference in height cannot emerge as a discontinuous jump. Figure 6.3 indicates how the jump may look, based on experimental observation for some values of the Froude number. In all cases the jump appears as a turbulent zone, with states of regular flow on both sides. The loss of mechanical energy as heat will equal the energy difference according to Eq. (6.15). The downstream liquid height must be in accord with downstream parameters like as for instance the slope. The jump will then take place at such values of h_1 and u_1 that Eq. (6.12) is satisfied.

A small localized disturbance in the surface height can be considered a wave with a large wavelength compared to the liquid height. Its propagation velocity relative to the liquid will

[2] In [Landau and Lifshitz 1987] this height level term is not added. That corresponds to referring the energy's reference level to the liquid surface. However, then one cannot compare the energy of two different cross-sections with different fluid heights but the same bottom level, which Landau and Lifshitz actually does.

[3] Here $b =$ constant, as different from gas flow in a pipe, where a variable pipe diameter makes possible a continuous variation in gas density on both sides of a shock.

Figure 6.3: Different types of hydraulic jump in a rectangular channel [Massey 1983]

then be given by Eq. (**??**) as $c = \sqrt{gh}$.[4] However, the stationarity implies that

$$\mathcal{F}_1 = \frac{u_1}{c_1} \quad , \quad \mathcal{F}_2 = \frac{u_2}{c_2} \tag{6.17}$$

and since $h_2 > h_1$ for a jump, Eqs. (6.10) and (6.11) imply that

$$\mathcal{F}_1 > 1 \quad , \quad \mathcal{F}_2 < 1 \tag{6.18}$$

We thus notice the analogy

- Froude number in liquid channel flow
  $\Leftrightarrow$
  Mach number in gas pipe flow

At the 'water shock', the disturbances which downstream have been fighting their way up against the flow's local 'subwave' velocity will be stopped, stamping against the flow racing with 'superwave' speed from upstream. In the same way as the transition from supersonic to subsonic velocity in gas flow can only take place at a shock, the hydraulic jump cannot be replaced by a smooth transition: Figuratively speaking, in both cases it is not possible for the *information* about downstream conditions to propagate upstream from the shock, so that downstream conditions can influence the upstream conditions. In practice, a hydraulic jump will often emerge near a change of slope in a channel. A change of the downstream slope can then influence the actual location of the jump, but not influence upstream conditions in toher ways.

6.1.1 Critical flow

Let us examine more closely the conditions at (an ideal) hydraulic jump. Using Eq. (6.6) we can rewrite (6.14) as

$$J_E = J_m g\, h_E \quad , \quad h_E = h + \frac{1}{2g}\,(\frac{J_m}{\rho b})^2 \frac{1}{h^2} \tag{6.19}$$

[4]The relation also follows simply from momentum conservation; see [Franzini and Finnemore 2002].

where h_E has dimension length (like the head). By various derivations of the expression for h_E we find:

$$(h_E)_{min} = \frac{3}{2}h_c \quad \text{for} \quad h = h_c = \left(\frac{(J_m/\rho b)^2}{g}\right)^{1/3} \quad (J_m \text{ constant}) \qquad (6.20)$$

$$(J_m)^2_{max} = g(\rho b)^2 h_c^3 \quad \text{for} \quad h = h_c = \frac{2}{3}h_E \quad (h_E \text{ constant}) \qquad (6.21)$$

The quantity h_c is called *critical height* (or depth). The corresponding critical flow velocity[5] and Froude number are:

$$u_c = \frac{J_m}{\rho b h_c} = \frac{(g h_c^3)^{1/2}}{h_c} = (g h_c)^{1/2} \qquad \Rightarrow \qquad \mathcal{F}_c = 1 \qquad (6.22)$$

Figure 6.4 illustrates these conditions for flow in a rectangular channel. We observe here still

Figure 6.4: Liquid depth as a function of energy head and mass flow rate [Massey 1983]

another analogy to adiabatic gas flow, *critical flow*. It corresponds to the behaviour in a *sonic* (or *critical*) nozzle, where the mass flow rate has a maximum if the flow velocity is sonic at the minimum diameter, and the gas will continue with supersonic velocity until downstream it will hit a standing shock wave:

- Maximum mass flow rate for a given h_E occurs at the height $h = h_c = \frac{2}{3}h_E$, with $\mathcal{F} = \mathcal{F}_c = 1$

If $h_E > (h_E)_{min} = \frac{3}{2}h_c$, the flow will take place in a *supercritical* ($\mathcal{F} > 1$) or in a *subcritical* ($\mathcal{F} < 1$) state, both with the same h_E, corresponding to the lower and upper branches of the curves, respectively. At critical conditions a comparatively large change in liquid depth will correspond to a comparatively small change in the energy head of the flow. Simultaneously, a surface disturbance propagating against the flow will become trapped. Thus, *standing waves* may be created, and if $h_E > \frac{3}{2}h_c$ a hydraulic jump may occur if the flow is supercritical. Figure 6.5 reminds us that the jump will always bring us to a state with a lower h_E.

[5] A misleading—but common—term. It is unrelated to the critical velocity for transition to turbulence as well as the critical thermodynamical state.

Figure 6.5: Liquid depth and energy head at a hydraulic jump [Massey 1983]

After a hydraulic jump has occurred, changes in the flow conditions downstream of the 'shock' will have no power to influence the upstream flow conditions. The analogy to flow in a critical nozzle is that between the minimum flow cross section and an eventual downstream shock wave the flow velocity will be supersonic. Changes in downstream flow conditions for a given incoming energy head will then have no possibility to communicate their existence to points upstream of the shock—thus, the maximum mass flow rate for a constant incoming h_E will occur for critical flow in the nozzle.

6.2 Solitary waves, briefly

The equations we have solved so far in these lecure notes, have been linearized versions of the nonlinear equations of motion. However, some particular solutions of simple nonlinear versions of the equations are known, and in some cases there are general transformations which make the equations completely integrable on analytical form [Bender and Orszag 1978]. In particular this applies to wave solutions, where the equation may change from partial to ordinary form by the transformation

$$u(x,t) = \varphi(\xi) \qquad , \qquad \xi = x - ct \tag{6.23}$$

The Scottish nobleman and physicist J. Scott Russell was among the first to realize that there are more kinds of waves between heaven and earth than those counted among the solutions of linearized equations [Hemmer 1994]:

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion: it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon ...

In the meantime there has been considerable progress in the understanding of the properties of such waves, based on solutions of nonlinear differential equations. We will mention a few examples to convey a feeling of what it is all about. References like [Hemmer 1994] can be recommended for those wanting to learn more about those matters.

A common terminology in the classification of waves is:

- A *permanent* wave is a wave propagating without changing its form
- A *solitary* wave is a localized permanent wave
- A *soliton* is a wave which asymptotically conserves its form and velocity in collisions with other solitary waves

Figure 6.6 shows examples of such waves. We will consider the properties of three nonlinear differential equations, and see how some of their solutions relate to such waves and to matters treated previously:

- The *free-flow equation*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{6.24}$$

- The *Burgers equation*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \tag{6.25}$$

- The *KdV equation*[6]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{6.26}$$

We recognize the first two as 1D special cases of the Navier-Stokes equation (**??**) for negligible pressure and field forces, respectively without and with the viscous term kept. The third emerges when the nonstationary Bernoulli equation on the complete form (**??**) is used to provide a nonlinear basis for the surface boundary condition for waves in liquids, instead of Eq. (**??**) (in the limit $\alpha \to 0$):

$$\rho g\zeta - \rho(\partial_t \phi)_{z=\zeta} + \frac{1}{2}(\nabla\phi)^2_{z=\zeta} = 0 \tag{6.27}$$

The KdV equation then emerges to lowest nontrivial order in simultaneous perturbation expansions in $(h/\lambda)^2$ and ζ_{max}/h [Hemmer 1994]. (As distinct from the flow velocity in the other two equations, u in Eq. (6.26) denotes the surface displacement.)

Insertion into the equations confirm that the first has a general implicit solution

$$u_{(6.24)} = f(x - u_{(6.24)}t) \tag{6.28}$$

with f an arbitrary function of its argument (to be determined by the initial condition), while the two others have the particular solutions

$$\varphi_{(6.25)}(\xi) = c - a + \frac{2a}{1 + e^{a(\xi - b)}} \tag{6.29}$$

$$\varphi_{(6.26)}(\xi) = \frac{3c}{\cosh^2\left[\frac{1}{2}\sqrt{c}(\xi - x_0)\right]} \tag{6.30}$$

[6]Shorthand of Korteveg–de Vries.

Figure 6.6: Various wave types [Hemmer 1994]. a) Periodic permanent wave. b) Solitary wave types (pulse and 'kink'). c) Two-soliton solution of a nonlinear differential equation, drawn at 3 instants ($t_1 < t_2 < t_3$)

Figure 6.7: Drawing of the a) Burgers and b) KdV wave solution [Hemmer 1994]

with a, b, c and x_0 being constants of integration. The latter two solutions represent permanent waves by definition. Figure 6.7 shows moreover that they are solitary waves, of the 'kink' and the pulse type, respectively.

To demonstrate the properties of solutions of the free-flow equation, and to understand better the properties of the two solitary wave solutions, it will be useful to consider the various terms more closely.

Nonlinearity: Let Eq. (6.24) denote the velocity field in a 1D jet with average velocity c, with a weak periodic perturbation superposed which will be assumed to be initially harmonic:

$$u = c + u'$$

$$\frac{\partial u'}{\partial t} + (c + u')\frac{\partial u'}{\partial x} = 0$$

Insertion of $u' \sim \cos(kx - \omega t)$ gives

$$V = \frac{\omega}{k} \approx c + u'$$

Figure 6.8: Temporal development of velocity field in a free jet [Hemmer 1994]

The phase velocity is seen to vary along the jet. The wave profile will become deformed with time: Those mass particles which already move faster than the average velocity ($u' > 0$) will be moving still faster, and those moving slower will increasingly slacken speed. This has been plotted in Figure 6.8, in a reference system moving with the average velocity c. The velocity profile will become steeper and steeper. Mathematically, multivaluedness will occur, as is indicated in the figure.[7] Which is of course physically impossible: *Shock fronts* will occur, as indicated by removal of the shaded parts of the curves; physically, *jumps* will develop.

Dissipation: If the second derivative in the Burgers equation (6.25) is neglected, the remaining nonlinearity will produce shocks of the type considered above. The second derivative will produce the opposite effect; remove instead the nonlinearity and you will obtain a diffusion equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Qualitatively, it will make sharp profiles become 'smeared out' with time,[8] as is drawn

[7] See for instance [Papatzacos 2003] for a more complete description of this phenomenon in terms of *characteristics*.

[8] The solutions of this equation are more adequately described in [Papatzacos 2003].

Figure 6.9: Discontinuous jump function under the influence of a diffusion equation

in Figure 6.9.

We may therefore interpret the solution (6.29), drawn qualitatively in Figure 6.7a, as an equilibrium between the tendency for nonlinear shock development and the dissipative front smearing. Such solutions are also often denoted as shocks.

Dispersion: Suppose that a given harmonic plane wave $u \sim \cos(kx - \omega t)$ has a k-dependent phase velocity, with the dispersion relation

$$\omega(k) = c_0 k - \beta k^3 \qquad (\beta > 0)$$

It can be easily checked by insertion that this dispersion relation follows from the wave equation

$$\frac{\partial u}{\partial t} + c_0 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0 \tag{6.31}$$

Such waves have a constant phase velocity in the limit of infinite wavelengths, and a weakly k-dependent phase velocity for shorter wavelengths. A wave which is a superposition of such plane waves, will usually become broadened with time due to the dispersion.

Eq. (6.31) can be considered as a linearized (and scaled) version of Eq. (6.26). That suggets the following interpretation of the pulse solution of the KdV equation: Eq. (6.30) and Figure 6.7b describe an equilibrium between nonlinear shock development and dispersive pulse smearing.

The result of Problem 6.3 is that a certain transformation of the dependent variable will bring the Burgers equation on a linearized form; a linearized diffusion problem emerges. It is therefore easy to construct solutions with several solitary waves of the type (6.29) simultaneosly present. Based on the diffusion representation one notices easily that coexisting single waves will not conserve their form and velocity after a collision; rather, they will combine into one new solution. Thus, the solutions of the Burgers equation are not solitons.

The KdV solution (6.30) can be written as

$$u(x,t) = 12 \frac{\partial^2}{\partial x^2} \ln U(x,t) \tag{6.32}$$

where

$$U(x,t) = 1 + f\ , \qquad f(x,t) = e^{-\sqrt{c}(x-x_0-ct)}$$

The transformation (6.32) does not produce a linear equation for U, but it is still useful in the search for solutions comprising *several* solitons [Hemmer 1994]. Let us cite the result for two superposed waves:

$$\begin{aligned} f &= f_1 + f_2 + f_1 f_2 \frac{(\sqrt{c_1} - \sqrt{c_2})^2}{(\sqrt{c_1} + \sqrt{c_2})^2}\ , \\ f_i &= e^{-\sqrt{c_i}(x-x_{0i}-c_i t)} \end{aligned} \tag{6.33}$$

This expression is an exact particular solution of Eq. (6.26) with the transformation (6.32) applied. For $t = t^* = (x_{02} - x_{01})/(c_1 - c_2)$ the separate solitary waves f_1 and f_2 will have the same maximum positions. In the limits $t \ll t^*$ and $t \gg t^*$ the expression (6.33) will represent the two separate solitary waves at large distance from each other, however with their order changed in the two limits—after the 'collision' each of the solitary pulses emerge again with unchanged form and velocity. Accordingly, we notice that the KdV equation possesses solutions which are solitons.

There is good reason to assume that J. Scott Russell was chasing a soliton!

6.3 Problems

Problem 6.1 Show that in the limit $\mathcal{F}_1 \gg 1$, the ratio between mechanical energy lost in a hydraulic jump and the incoming energy will approach 1.

Problem 6.2 Water flows with the rate $5.4\ \mathrm{m^3\,s^{-1}}$ under a sluice gate in a channel with rectangular cross-section and width 3.5 m, resulting in a water depth of 0.38 m. A hydraulic jump occurs downstream of the sluice position. Calculate the water depth downstream of the jump, as well as the mechanical power dissipated during the jump.

Problem 6.3 Show that the Burgers equation becomes linearized by the *Cole-Hopf transformation*

$$u = -2\frac{\partial}{\partial x} \ln T$$