Chapter 1

Introduction

We start by summarizing some basic equations and derived relations which will be assumed known from previous courses. After that, a justification of the contents and objective of this course will follow.

1.1 Recapitulation

1.1.1 Notation and vector relations

In this course we will treat fluids which will be assumed to be

- *one-phase*,
- *incompressible* (unless otherwise indicated explicitly), as well as
- *Newtonian* (viscosity function independent of the angular deformation velocity in the fluid [Irgens 1983]);

accordingly the density ρ as well as the dynamical and kinematical viscosities μ and $\nu\,(=\mu/\rho)$ will be treated as constants.

The SI system is being used in the compendium. In cases where the notation varies between common textbooks, the notation of [Tritton 1988] is mostly used.

Fluid dynamics is based on the assumption that a description in terms of *fields* is valid for the local velocity $\boldsymbol{u}$ and pressure p, that is, $\boldsymbol{u} = \boldsymbol{u}(x, y, z, t)$ and $p = p(x, y, z, t)$ in cartesian coordinates. A more compact notation which does not presuppose anything about the type of coordinate system, is $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r}, t)$, where $\boldsymbol{r}$ is the position vector. The alternative cartesian notations

$$\boldsymbol{r} = (x, y, z) = (x_1, x_2, x_3) \tag{1.1}$$

$$\boldsymbol{u} = (u, v, w) = (u_1, u_2, u_3) \tag{1.2}$$

will be used, as well as a compact notation for the partial derivation operator:

$$\partial_i \equiv \frac{\partial}{\partial x_i} \quad (i = 1, 2, 3)\ , \qquad \partial_t \equiv \frac{\partial}{\partial t} \tag{1.3}$$

The vector operator ∇ (*del* or *nabla*) is defined in cartesian coordinates as

$$\begin{aligned}\nabla &= \delta_{ij}\hat{\boldsymbol{e}}_i\partial_j \\ &= \hat{\boldsymbol{e}}_i\partial_i \\ &= \hat{\boldsymbol{e}}_1\partial_1 + \hat{\boldsymbol{e}}_2\partial_2 + \hat{\boldsymbol{e}}_3\partial_3 \end{aligned} \tag{1.4}$$

where $\hat{\boldsymbol{e}}_i$ $(i = 1, 2, 3)$ are orthogonal unit vectors, and the *Kronecker delta* δ_{ij} is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{1.5}$$

Equation 1.4 is an example of a convention we will be using consistently in this course, to avoid writing the summation symbol:

- *Summation convention*: For pairs of identical latin indices in products of cartesian coordinates, a sum over the index values from 1 to 3 is implied

The operator appears in the expressions

$$\operatorname{grad}\Phi \equiv \nabla\Phi\,, \qquad (\operatorname{grad}\Phi)_i = \partial_i\Phi \quad (i = 1, 2, 3) \tag{1.6}$$

$$\operatorname{div}\boldsymbol{u} \equiv \nabla\cdot\boldsymbol{u} = \partial_i u_i \tag{1.7}$$

$$\operatorname{curl}\boldsymbol{u} \equiv \nabla\times\boldsymbol{u}\,, \qquad (\operatorname{curl}\boldsymbol{u})_i = \epsilon_{ijk}\partial_j u_k \tag{1.8}$$

where Φ is a scalar field, $\boldsymbol{u}$ a vector field, and ϵ_{ijk} is the *Levi-Civitá tensor* which allows us to write components of vector products in a compact way. It is defined by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for } (ijk) \text{ cyclic permutation of } (123) \\ -1 & \text{for } (ijk) \text{ anticyclic permutation of } (123) \\ 0 & \text{otherwise} \end{cases} \tag{1.9}$$

We will need the general relations

$$\nabla\times(\nabla\Phi) \equiv 0 \tag{1.10}$$

$$\nabla\cdot(\nabla\times\boldsymbol{u}) \equiv 0 \tag{1.11}$$

$$\nabla\times(\nabla\times\boldsymbol{u}) = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla^2\boldsymbol{u} \tag{1.12}$$

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \tfrac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) \tag{1.13}$$

$$\nabla\times((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) = (\boldsymbol{u}\cdot\nabla)(\nabla\times\boldsymbol{u}) + (\nabla\cdot\boldsymbol{u})(\nabla\times\boldsymbol{u}) - ((\nabla\times\boldsymbol{u})\cdot\nabla)\boldsymbol{u} \tag{1.14}$$

which are derived in Appendix A by methods which are also valid for more complicated products. The scalar multiplication symbol and the parentheses are customarily deleted in cases where the meaning is obvious.

The operations grad, div, curl og ∇^2 (the *Laplace operator*) can of course also be expressed in curvilinear coordinates. We will not present the derivations of any such expressions, many of which are complicated. In Appendix A, however, some particular expressions in cylindrical and spherical polar coordinates are listed; we will need them in later chapters. A more complete presentation can be found in formula collections like [Rottmann 1995], and in the textbooks in fluid dynamics from the series in theoretical physics mentioned in the list of references.

1.1.2 Equations and concepts for fluid flow

In this section we will summarize the general case of a compressible fluid. In later chapters we will implicitly specialize to incompressible flow, unless otherwise mentioned.

The subject is assumed to have been presented to the students in basic textbooks with an emphasis on engineering applications, for instance [Olson 1980], [Gerhart et al. 1992], [Shames 1992] or [White 1999], or less completely concerning equations of motion in [Franzini and Finnemore 2002] or [Massey 1983]. Lecture notes and short texts like [Palm 1977] or [Ytrehus 1981] can supplement the presentation in the latter two.

To find the acceleration of a fluid particle, or the variation with time of any property of the material substance following the flow, a derivative along the particle's path of motion:

- The *substantial derivative*

$$D_t = \partial_t + (\boldsymbol{u}\cdot\nabla) \tag{1.15}$$

The equation of motion has to be combined with an equation expressing that the flowing mass is conserved:

- The *continuity equation*

$$\partial_t\rho + \nabla\cdot(\rho\boldsymbol{u}) = 0 \tag{1.16}$$

For an incompressible fluid ($\rho =$ constant) the continuity equation wil reduce to the condition

$$\nabla\cdot\boldsymbol{u} = 0$$

In the general compressible case, another equation is needed which gives the connection between the density and the other thermodynamical variables:

- The *equation of state*, for instance for a gas

$$f(\rho, p, T) = \text{constant} \tag{1.17}$$

(In the incompressible case ($\approx$ for a liquid) there is also an equation of state, namely,

$$\rho = \text{constant} \quad)$$

Streamlines are continuous curves with the velocity vector in a point tangential to the streamline through that point. With $d\boldsymbol{s} = (dx, dy, dz)$ an infinitesimal path element along the line, the condition $\boldsymbol{u}\times d\boldsymbol{s} = 0$ gives two equations determining the expressions for the streamlines:

- The *streamline equations*

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{1.18}$$

Assume that the continuity equation for a two-dimensional stationary incompressible flow is satisfied. Then a function Ψ can be defined, with the property $d\Psi = 0$ along a streamline, which determines the components of the velocity:

- Stream velocity expressed by the *stream function* Ψ (in 2D only)

$$u = \frac{\partial \Psi}{\partial y} , \qquad v = -\frac{\partial \Psi}{\partial x} \tag{1.19}$$

The rotational property of a flow can be expressed by one local and one macroscopical quantity, related to each other by Stokes's law (Appendix A, Eq. (A.2)):

- The *vorticity* $\boldsymbol{\omega}$

$$\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u} \tag{1.20}$$

- The *circulation* Γ along a closed curve L, located at the boundary of an enveloped surface S

$$\begin{aligned} \Gamma &= \oint_L \boldsymbol{u} \cdot d\boldsymbol{l} \\ &= \int_S \boldsymbol{\omega} \cdot d\boldsymbol{S} \end{aligned} \tag{1.21}$$

Here, $d\boldsymbol{l}$ is a differential tangent vector to the curve, and $d\boldsymbol{S} = \hat{\boldsymbol{n}}\, dS$, $\hat{\boldsymbol{n}}$ being a perpendicular unit vector in a point on the surface. A zero vorticity flow is called *irrotational*. For irrotational flow of an incompressible fluid, Eq. (1.10) implies that the velocity can be expressed as the gradient of a potential (with some textbooks using the opposite sign convention):

- The *velocity potential* Φ

$$\boldsymbol{u} = -\operatorname{grad} \Phi \tag{1.22}$$

If the flow also satisfies the continuity equation (1.16), that implies a second order differential equation for an incompressible flow:

- The *Laplace equation*, valid also for a nonstationary flow

$$\nabla^2 \Phi \;=\; \partial_i \partial_i \Phi \;=\; 0 \tag{1.23}$$

This equation has been the starting point for a large number of studies of fluid dynamical problems, in particular in a two-dimensional context.

The dynamical aspects of the flow are introduced by two equations which both embody Newton's second law for a flowing element of fluid:

- *Euler's equation*, compressible ideal (nonviscous) flow

$$\rho D_t \boldsymbol{u} = -\nabla p + \rho \boldsymbol{g} \tag{1.24}$$

- The *Navier-Stokes equation* (3 components), incompressible real (viscous) flow

$$\rho D_t \boldsymbol{u} = -\nabla p + \rho \boldsymbol{g} + \mu \nabla^2 \boldsymbol{u} \tag{1.25}$$

In Eqs. (1.24) and (1.25) it is assumed that the gravity $\rho\boldsymbol{g}$ is the only external field force, $\boldsymbol{g}$ being the acceleration of gravity. The derivation of the viscous term in Eq. (1.25) is summarized in Appendix B, where it is also shown that addition of a term

$$(\zeta + \frac{\mu}{3})\ \nabla(\nabla \cdot \boldsymbol{u})$$

at the right-hand side gives the corresponding equation for the compressible case. ζ is called the *second coefficient of viscosity* (μ being the first).

Altogether, the equation of motion, the continuity equation and the equation of the state give 5 equations for the determination of the 5 unknowns u_i, p og ρ.

For a stationary flow, the Euler equation gives an expression which can be integrated along a streamline. With z the vertical coordinate, $q = |\boldsymbol{u}|$, and $g = |\boldsymbol{g}|$:

- The *Bernoulli equation*, ideal incompressible fluid, along a streamline

$$\frac{1}{2g} q^2 + \frac{p}{\rho g} + z = \text{constant} \tag{1.26}$$

For a polytropic gas (for instance an adiabatical ideal gas, see Appendix C) the same expression is obtained, only with the pressure term multiplied by $\kappa/(\kappa - 1)$, where κ is the polytropic exponent. Bernoulli's equation corresponds to the energy equation in solid body dynamics. Each term in (1.26) represents dimensionally a height (*head*).

Similarity between flow cases described by the Navier-Stokes equation, depends on the values of dimensionless parameters:

- The *Reynolds number* (U and L characteristic scales of velocity and length

$$\mathcal{R} = \frac{\rho U L}{\mu} = \frac{\text{inertial force}}{\text{viscous force}} \tag{1.27}$$

- The *Froude number*

$$\mathcal{F} = \frac{U}{\sqrt{gL}} = \left(\frac{\text{inertial force}}{\text{gravity force}}\right)^{1/2} \tag{1.28}$$

- The *Mach number*

$$\mathcal{M} = \frac{U}{c} = \frac{\text{velocity of flow}}{\text{velocity of sound}} \tag{1.29}$$

Two different types of flow appear:

- *Laminary* vs. *turbulent* flow; one regular flow where the fluid layers "slide" past each others, as different from the other, an irregular and fluctuating flow, for low and high values of the Reynolds number, respectively (the transition taking place at $\mathcal{R} \sim 2200$ for pipe flow).

We will see later that this mental picture from the elementary courses is an oversimplification.

Figure 1.1: Velocity vector and normal vector at a wall

1.1.3 Boundary conditions

The differential equation has to be supplemented with boundary conditions. In this course we will only consider the case of a rigid and impenetrable wall. With flow separation at the wall disregarded, and $\hat{\boldsymbol{n}}$ denoting a local surface normal in a point on the wall, one boundary condition is obtained which holds both for viscous and nonviscous flow at a wall (se Figur 1.1):

- The *impermeability condition*

$$\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = 0 \tag{1.30}$$

For a *viscous* fluid, with the experimentally confirmed assumption that the friction against the wall is of the same nature as between two fluid layers, without any discontinuity, an additional condition at the wall is obtained:

- The *no-slip condition*

$$\boldsymbol{u}\times\hat{\boldsymbol{n}} = 0 \qquad \text{(viscous fluid)} \tag{1.31}$$

This condition has to be used together with the Navier-Stokes equation. It is invalid in the limit where the mean free path of the molecules is of the same order as the fluid's geometric dimension, however in that case neither a continuum description in terms of fields holds.

The boundary condition at infinity, if such is needed, is of the type

$$u \to u_0 \qquad \text{for} \qquad r \to \infty \tag{1.32}$$

The Laplace equation together with the Neumann condition (1.30) gives a *solvable* problem for Φ up to an additive constant (see for instance [Papatzacos 1991]). If in addition Eq. (1.31) is introduced, an *overdetermined* system without a solution is obtained. Thus there is seemingly the paradox that a large number of important flow problems are solvable with sufficient accuracy by potential theory, where one is precluded from introducing the physically correct no-slip condition for a real fluid. In a later chapter on flow regimes we will see that the *boundary layer* concept solves this paradox.

1.2 The contents and purpose of the course

As a background for a motivatian for the contents of the course, we will first set up qualitatively a status for the mathematical side of fluid dynamics.

All experience indicates that the Navier-Stokes equation and its compressible counterpart, Eq. (B.16), describes practical fluid dynamics adequately. That holds for flow and fluid conditions where the assumption about a Newtonian fluid is correct. However, the Navier-Stokes equation, as well as its ideal flow companion, the Euler equation, is *nonlinear* due to the convective acceleration term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$. This condition renders the equations hard to solve analytically; in fact it is only for a minority of dynamical situations that the flow can be described by such an exact solution of the equation of motion. Simplifying assumptions are usually the basis of the existing analytical solutions, with *stationary flow* as a usual common denominator. It is characteristical for a large class of the solutions that the nonlinear term plays a secondary role or has no influence at all in the equation. For the extreme simplification of *irrotational* flow we can apply *potential theory*, where the supertuned method apparatus of mathematics lies ready to attack the problems and find solutions in a systematic way. The downside is that such simplifications often result in solutions which do not represent real flows adequately.

The Navier-Stokes and Euler equations can of course also be solved numerically and with all terms kept. The development of supercomputers has resulted in the possibility of modeling fluid flow far beyond the limitations of analytical methods. Our belief in the Navier-Stokes equation as the "right one" for a Newtonian fluid, stems to a large extent from the comparison of such solutions with model experiments and observations. One publicly wellknown supercomputer solution of the equation of motion of a fluid is known as *long-term weather forecast*. Surely, the range of numerical solution methods will continue to grow, both for macro- and microsolutions of fluid flow.

Our final conquest of the solutions will however be behind the horizon forever. That is due to *sensitive dependence of the boundary conditions*, which is characteristical for a large class of nonlinear equations: Small deviations in the initial conditions, even the round-off error in the representation of numbers, give rise to exponentially growing deviations in the numerical solutions. The reliability of long-term weather forecasts is a useful illustration, where the progress in avoiding the equivalence of pure guesses is typically 1–2 days at a time, irrespectively of the magnitude of the progress in computer technology and algorithms.[1]

It may therefore look as if a physical understanding of fluid dynamics will continue to be a condition, not only for those working within the discipline, but also for those having a relation to its applications. The ability to find solutions by making smart approximations will have its place both in addition to and as a basis for numerical calculations.

After a continuation of fluid dynamical basis relations and a discussion of their conditions of validity, we will therefore study some selected main topics, with an emphasis on the physical

[1]According to a popularized presentation [Gleick 1987], a few years ago some millions of US$ were spent on "weather control", of course to no avail in the end. The knowledge of the principial limitations of the solution methods was available at the time, however apparently not to the right persons. No guarantee for comparably large savings is attached to these lecture notes, however the practical relevance of an understanding of the physical contents of fluid dynamics should not be underestimated!

background for approximations and solutions:

Viscous flow
Potential flow (free-surface case)
A shock wave analogy
Laminar boundary layers
Flow separation, instabilities, transition to turbulence (qualitatively)
Turbulence: Semiempirical description
Turbulence: Statistical description

(Thermoconvection will not be treated; that would swamp the time frame for a 5 study points course.) The main themes will be illustrated through applications where the methods of solution are covered in detail. It is attempted to present generally applicable methods.

The order of presentation reflects the historical development of fluid dynamics to some extent. Turbulence, observed and defined 120 years ago, turned out to be a particularly obstinate speciment of hard-to-solve problems. Only the last half-century has seen a development considerably past a semiempirical description. A simple introduction to the recent statistical theories is attempted in these lecture notes. Hopefully, the notes may serve as an introduction to further study of this or other topics.

1.3 Problems

For a revival of dormant abilities:

Problem 1.1 Show that the continuity equation for a compressible fluid can be rewritten as

$$D_t\rho + \rho\nabla\cdot\boldsymbol{u} = 0$$

Problem 1.2 a) Find the stream function for the following two velocity potentials, provided the flow is incompressible:

$$1. \qquad \Phi = -\,C\,(x^2 + y^2)$$
$$2. \qquad \Phi = -\,C\,(x^2 - y^2)$$

b) Find the velocity potentials for the following two stream functions, provided the flow is irrotational:

$$1. \qquad \Psi = C\,(x^2 + y^2)$$
$$2. \qquad \Psi = C\,(x^2 - y^2)$$

Sketch the streamlines for positive C, as well as equipotential lines, if possible.