Chapter 2

Continuation of the basics

Many of the topics in this chapter may be considered as elementary as those in the preceding one. Due to limitations of time and scope in elementary courses for engineering students, however, these topics are often treated in a shallow way there, if at all.

2.1 Streamlines, particle paths, and streaklines

A reminder:

- *Streamlines* are instantaneous curves through the fluid, with the local velocity vector as a tangent in all points
- *Particle paths* are curves describing a fluid particle's position as a function of time
- *Streaklines* are curves created by continuous injection of for instance dye in a fixed point in space in the flowing fluid

For a stationary flow in a fixed outer constraint, the streamline, the particle path and the streakline through a fixed point will be identical. For a nonstationary flow that is not the case.

As an illustration, consider a 2D stationary flow perpendicular to the axis of a fixed and infinitely long cylinder. We can also choose to describe this flow situation in a reference frame moving with the velocity of the fluid far from the cylinder, such that the fluid at infinity is "at rest" while the cylinder is moving; we would then observe a nonstationary flow. The following figures show the results for a *computed* stream with $\mathcal{R} = 40$ [Tritton 1988]. Such a stream is thus laminar. A detailed experimental check of the calculational results would be complicated since low $\mathcal{R}$ have small length scales as an experimental condition, but the main features coincide with known experimental results.

Figure 2.1 shows streamlines for a stationary flow wiht a constant velocity $\boldsymbol{u}_0$ (towards the right) at a large distance from the fixed cylinder. Only the upper half of the flow has been shown because of the symmetry. The closed streamlines correspond to a stationary eddy.[1]

The following figures have the common trait that a fixed term has been subtracted from the velocity vector in all points:

$$\boldsymbol{u} = \boldsymbol{u}_{\text{Fig.2.1}} - \boldsymbol{u}_0 \tag{2.1}$$

[1]At this point we will neglect that these advanced calculations for a laminary flow also show the emergence of flow separation and eddies—such phenomena will be treated in later chapters.

Figure 2.1: Stationary flow past a circular cylinder, $\mathcal{R} = 40$

This corresponds to the case where the cylinder moves with velocity $-\boldsymbol{u}_0$ (towards the left) through a fluid which is at rest at infinity. Figure 2.2 shows the emerging streamline pattern. If the frame in the figure could have been enlarged without bounds, one would observe that all streamlines are either closed or start and end on the cylinder. Which does not imply that the fluid particles are moving along such closed paths: The figure is an *instantaneous* picture of the flow while the cylinder is located in the middle of the frame. The whole picture is moving to the left with a constant speed $-\boldsymbol{u}_0$.

Figure 2.3 shows particle paths in the reference frame where the cylinder is moving. In part a) the letters A, B and C refer to the corresponding positions in Figure 2.2. When a fluid particle has reached point A, the cylinder is thus just below it in the figure. The paths in part b) correspond to particles which originally had larger distances from the plane of symmetry. They have been drawn in such a way that each particle is located just above the cylinder at the same time (in the point denotes by a cross) at the same time. All the paths have finite lengths, which in reality are somewhat larger than shown, since the figure comprises a finite interval in time.

The paths for particles in the stationary eddy in Figure 2.1 is an exception, since they will be transported without bounds together with the cylinder. Figure 2.4, compressed a factor 20 in the $\boldsymbol{u}_0$ direction, shows the paths for a few such particles, located in the points marked by crosses when the cylinder is at the indicated position. The wavy form is due to the movement in an eddy. That some paths are crossing each others, means that the particles pass the indicated crossing points at different times.

The solid curves in Figure 2.5 show two streaklines for a continuous discharge from point S. The curve SA shows a discharge starting a given time interval before the cylinder reached the indicated position. When the cylinder moves on (still with dye being discharged), the dye will after an equally long time interval later form the curve SBC. The broken curves show two particle paths. SAC indicates the path for the first particle when the discharge started, SB indicates the path of a particle ejected when the cylinder was at the position shown.

Figure 2.2: Streamlines at a cylinder moving through a fluid at rest

Figure 2.3: Particle paths for a cylinder moving through a fluid at rest

Figure 2.4: Paths for particles in the wake behind the cylinder

Figure 2.5: Streaklines and particle paths (see the text)

As indicated, the relations between the flow phenomena in such visualization experiments can be quite complex. Their interpretation may require care and forethought. In a later chapter we will mention an experiment where streaklines were used, in connection with instability of boundary layers.

2.2 Bernoulli's equation for a nonstationary potential flow

The usual form of Bernoulli's equation for "technical purposes", Den vanligste formen av Bernoullis likning til "teknisk bruk", Eq. (**??**), was derived for the following conditions:

- Stationary flow
- Incompressible fluid
- Conservative (i.e. irrotational) mass force field, so that $\boldsymbol{g}$ can be expressed by the gradient of a potential $\mathcal{U}$ denoting potential energy per unit mass:

$$\boldsymbol{g} = -\operatorname{grad} \mathcal{U} \tag{2.2}$$

No assumptions were made about the velocity field. Therefore, the constant in (**??**) does not have to be the same for all streamlines. It turns out that we can relax the requirement of stationarity, and still find a conserved quantity in the timedependent flow, provided we make the additional assumption that the velocity can be expressed as the gradient of a potential according to Eq. (**??**):

- Irrotational flow

We start by the Euler equation (**??**), and insert from (**??**)in the convective part of the substantial derivative, as well as from Eq. (2.2):

$$\partial_t \boldsymbol{u} + \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = -\frac{\nabla p}{\rho} - \nabla\mathcal{U} \tag{2.3}$$

The third term on the LHS will disapper for a potential flow ($\boldsymbol{\omega} = 0$). We insert for $\boldsymbol{u}$ from Eq. (**??**) in the remaining terms on the LHS. In the first term we permute the order of the two derivations of Φ, so that all the terms can be expressed by gradients:

$$\nabla\,(-\partial_t\Phi + \frac{1}{2}(\nabla\Phi)^2 + \frac{p}{\rho} + \mathcal{U}) = 0 \tag{2.4}$$

When the gradient of some quantity is zero, this quantity must be a constant. Since $\boldsymbol{g} = (0, 0, -g)$, we can integrate the z component of Eq. (2.2) and get $\mathcal{U} = gz + \tilde{f}(t)$. Insertion into (2.4), with subsequent integration, og integrasjon, gives another t-dependent integration constant into which $\tilde{f}(t)$ can be included:

- *Bernoulli's equation for nonstationary potential flow*:

$$-\frac{1}{g}\,\partial_t\Phi + \frac{1}{2g}\,(\partial_i\Phi)^2 + \frac{p}{\rho g} + z = f(t) \tag{2.5}$$

However, we may choose $f(t) = \text{konstant}$, since a replacement of Φ $\Phi - g\int f(t)\,dt$ gives the same physical velocity vector $\boldsymbol{u}$. As distinct from Eq. (**??**), this arbitrary constant is the same for all streamlines.

In this course we will use a linearized version of Eq. (2.5) as a starting point for the study of surface waves in a liquid.

2.3 Vorticity and circulation

Those quantities, defined by Eqs. (**??**) an (**??**), give a local and a global measure, respectively, of a fluid's rotational properties. We will first consider the vorticity for some explicitly given flow fields. We then go on to derive an equation of motion for vorticity, analogous to the Navier-Stokes equation for $\boldsymbol{u}$. Finally we will prove a conservation theorem for the circulation in a nonviscous fluid.

2.3.1 Examples of flow fields with vorticity and circulation

"Solid-body" rotation: This type of flow is defined by

$$\boldsymbol{u} = \boldsymbol{\Omega} \times \boldsymbol{r} \tag{2.6}$$

where $\boldsymbol{\Omega}$ is the rotational angular frequency vector. Using relations from Chapter 1 and Appendix A we get:

$$\begin{aligned} \omega_i &= \epsilon_{ijk}\partial_j \epsilon_{mnk}\Omega_m x_n \\ &= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\Omega_m \delta_{jn} \\ &= (\delta_{im}\delta_{jj} - \delta_{ij}\delta_{jm})\Omega_m \\ &= 2\,\Omega_i \end{aligned} \tag{2.7}$$

A velocity field with this rotational property has a constant vorticity in all points, twice the angular frequency. The circulation around a circular curve in a plane with Ω as a normal vector will be proportional to the circle radius squared, according to Eq. (**??**).

Line vortex: [2] Consider the following flow field in cylindrical polar coordinates (see Appendix A), with K a constant:

$$u_\phi = \frac{K}{r} \quad , \quad u_r = u_z = 0 \tag{2.8}$$

It describes a rotation around the z axis, with rotational velocity inversely dependent on the distance from the axis. Because of the rotational symmetry (instead of going through a detailed calculation) we see that the vorticity vector cannot have any componenet perpendicular to the z axis:

$$\omega_r = \omega_\phi = 0 \tag{2.9}$$

Using Eq. (**??**) we get the third component:

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r\,u_\phi) = 0 \qquad (r \neq 0) \tag{2.10}$$

This flow field has zero vorticity in all points not on the z axix (for $r = 0$ neither the velocity nor the vorticity is defined). Still, the circulation around any closed curve around the z axis is nonzero. Since $\boldsymbol{\omega} = 0$ for $r > 0$, the curve can be deformed into a circle in the xy plane with center on the z axis, without any change of the circulation:

$$\Gamma = \int_0^{2\pi} u_\phi\, r\, d\phi = 2\pi K = \text{konstant} \tag{2.11}$$

[2] Alternatively called *free vortex* or *irrotational vortex*.

Figure 2.6: Successive positions for a volume element in a) solid-body rotation and b) circulation with zero vorticity

This is related to the singularity on the z axis. In the chapter on lift we will see that the concept of vorticityfree circulation is essential for an airplane's ability to stay in the air. flight of an airplane.

Shear flow: Consider a velocity field for a flow in the x direction, with the magnitude of the velocity dependent on y only:

$$u = u(y) \qquad , \qquad v = w = 0 \tag{2.12}$$

A calculation of $\nabla \times \boldsymbol{u}$ gives:

$$\omega_x = \omega_y = 0 \tag{2.13}$$

$$\omega_z = -\frac{\partial u}{\partial y} \tag{2.14}$$

Unless $u(y) =$ constant, we get a vorticity component in the z direction.[3]

The first two examples serve to indicate that circulation is due to a *change of direction*, and not to the closed path. Figure 2.6 shows an analogy, where case a) has vorticity, case b) not. The third example shows that circulation can be due to *shear deformation* of volume elements in the flowing fluid. Figure 2.7 shows examples of deformation with and without such circulation.

Section B.2 of Appendix B shows how an arbitrary deformation of a volume element can be splitted into terms with and without vorticity.

2.3.2 The vorticity equation and vortex lines

We will derive an equation for the development in time of the vorticity in a viscous and incompressible fluid. The mass force field will be assumed to be conservative, as in Eq. (2.2). We start by letting the curl operator act on both sides of the Navier-Stokes equation (**??**). Because of the relation (**??**) the contributions from pressure and mass force will disappear. In the rest of the terms we may change the order of the derivations, so that curl acts prior to ∂_t and ∇^2. Using Eq. (**??**) we may write the result as:

$$\partial_t \boldsymbol{\omega} + \nabla \times ((\boldsymbol{u} \cdot \nabla) \boldsymbol{u}) = \nu \nabla^2 \boldsymbol{\omega} \tag{2.15}$$

[3]This result, applied in a later chapter, will make us realize that even laminar boundary layers adjacent to potential flows will have lots of vorticity.

Figure 2.7: Volume elements in a) shear flow and b) rotationfree deformation

In the second term on the RHS we make an insertion from Eq. (**??**). In the latter equation, the second term on the RHS equals zero for an incompressible fluid, so that:

$$\partial_t \boldsymbol{\omega} + (\boldsymbol{u}\cdot\nabla)(\nabla\times\boldsymbol{u}) = ((\nabla\times\boldsymbol{u})\cdot\nabla)\boldsymbol{u} + \nu\,\nabla^2\boldsymbol{\omega} \tag{2.16}$$

Eq. (**??**) applied once more, and a reintroduction of the substantial derivative, gives

- the *vorticity equation*:

$$D_t\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} + \nu\,\nabla^2\boldsymbol{\omega} \tag{2.17}$$

On component form:

$$\partial_t\omega_i + u_k\partial_k\omega_i = \omega_m\partial_m u_i + \nu\partial_n\partial_n\omega_i \tag{2.18}$$

Compare now the equation to the Navier-Stokes equation and to the usual scalar diffusion equation, and the interpretation of the equation follows:

- The second term on the RHS lets the vorticity diffuse down along a 'vorticity gradient'; the first term on the RHS represents the influence of velocity changes

As the meaning of the first term is least evident, let us consider the equation for the nonviscous case $\nu = 0$. We write

$$\boldsymbol{\omega} = (\xi, \eta, \zeta) \tag{2.19}$$

but will consider a case where $\boldsymbol{\omega}$ initially is parallel to the z axis ($\xi = \eta = 0$):

$$D_t\boldsymbol{\omega} = \zeta\frac{\partial\boldsymbol{u}}{\partial z} \tag{2.20}$$

We will need to consider 2D and 3D flows separately.

2D flow: Suppose that $\boldsymbol{u} = \boldsymbol{u}(x, y)$, as well as $w = 0$ or constant. The condition $\xi = \eta = 0$ is then trivially satisfied, and the vorticity of a fluid particle in the flow is *conserved*:

$$D_t\boldsymbol{\omega} = 0 \tag{2.21}$$

Figure 2.8: Two successive positions of a vortex line with two fluid particles

3D flow: In this case $\boldsymbol{u} = \boldsymbol{u}(x, y, z)$. On component form the equation reads:

$$D_t\xi = \zeta\frac{\partial u}{\partial z} \tag{2.22}$$

$$D_t\eta = \zeta\frac{\partial v}{\partial z} \tag{2.23}$$

$$D_t\zeta = \zeta\frac{\partial w}{\partial z} \tag{2.24}$$

The first two components describes *vortex twisting*, since a variation of u and v in the z direction makes $\boldsymbol{\omega}$ change its direction. The third term represents *vortex stretching* with an accompanying change of strength: A variation of w with z will stretch or shorten the fluid particle in the z direction (with an accompanying change of cross section in the xy plane because of continuity), with the rotation simultaneously becoming faster or slower. Notice the analogy to solid body mechanics, where in an example the renowned dancer pulls the arms towards the body during a pirouette and then rotates faster.

The terms vortex stretching and vortex twisting can also be given an interpretation for viscous flow ($\nu \neq 0$). In a given situation with $D_t\boldsymbol{\omega} = 0$, one may talk about a balance between for instance vortex twisting and viscous diffusion of vorticity.

Analogously to streamlines, we may now define:

- *Vortex lines* are instantaneous curves through the fluid, with $\boldsymbol{\omega}$ as tangent in all points

Theorem:

- For a nonviscous flow ($\nu = 0$) a given vortex lines will always pass through the same fluid particles

The theorem can be proved by considering two particles on a vortex line, with infinitesimal distance ϵ and $\boldsymbol{\omega} = (0, 0, \zeta)$ at $T = 0$. After a time δt the line will have changed its position, see Figure 2.8:

$$\boldsymbol{\omega}_{t=\delta t} = (\zeta\frac{\partial u}{\partial z}\delta t,\ \zeta\frac{\partial v}{\partial z}\delta t,\ \zeta + \zeta\frac{\partial w}{\partial z}\delta t) \tag{2.25}$$

At $t = 0$ the velocity is (u, v, w) i punkt A, while in point B it is

$$(u + \frac{\partial u}{\partial z}\epsilon,\ v + \frac{\partial v}{\partial z}\epsilon,\ w + \frac{\partial w}{\partial z}\epsilon) \tag{2.26}$$

Figure 2.9: Two successive positions of a material loop

Then there are the following position differance vectors between the points:

$$AA' = (u\delta t,\ v\delta t,\ w\delta t) \tag{2.27}$$

$$BB' = ((u + \frac{\partial u}{\partial z}\epsilon)\delta t,\ (v + \frac{\partial v}{\partial z}\epsilon)\delta t,\ (w + \frac{\partial w}{\partial z}\epsilon)\delta t) \tag{2.28}$$

$$A'B' = (\frac{\partial u}{\partial z}\epsilon\delta t,\ \frac{\partial v}{\partial z}\epsilon\delta t,\ \epsilon + \frac{\partial w}{\partial z}\epsilon\delta t) \tag{2.29}$$

A comparison of the expressions (2.25) and (2.29) shows that the infinitesimal line element between the points will have the same direction as $\boldsymbol{\omega}$ at all times. Since this holds for all particle pairs with an infinitesimal interdistance, the theorem has been proved.

2.3.3 The Kelvin-Helmholz circulation theorem

Consider a closed loop which at all times pass through the same fluid particles in a flow (a *material loop*), and is thus following the flow; see Figure 2.9. Let the flow be compressible in general, and assume that the flow is isentropic and that the mass force field is conservative. Then the *Kelvin-Helmholz circulation theorem* claims that:

- If the flow is nonviscous, then the circulation around the loop is conserved:

$$D_t \oint \boldsymbol{u}\cdot d\boldsymbol{l} = 0 \tag{2.30}$$

Here, D_t is interpreted as an operation in all points on the material loop, while the integration around the loop is instantaneous. The proof involves an interchange of the order of integration and substantial derivation:

$$D_t \oint \boldsymbol{u}\cdot d\boldsymbol{l} = \oint (D_t\boldsymbol{u})\cdot d\boldsymbol{l} + \oint \boldsymbol{u}\cdot D_t(d\boldsymbol{l}) \tag{2.31}$$

This is allowed also for the convective part, because a derivation in time by the chain rule is involved. In the first term, introduce Euler's equation (**??**) and Eqs. (2.2) and (**??**):

$$\begin{aligned} \oint (D_t\boldsymbol{u})\cdot d\boldsymbol{l} &= -\oint (\frac{\nabla p}{\rho} + \nabla\mathcal{U})\cdot d\boldsymbol{l} \\ &= -\oint d(h + \mathcal{U}) \\ &= 0 \end{aligned} \tag{2.32}$$

In the second term, notice first that $\boldsymbol{dl}$ is a difference between two position vectors $\boldsymbol{r}$, while $D_t\boldsymbol{r} = \boldsymbol{u}$:

$$\begin{aligned} \oint \boldsymbol{u}\cdot D_t(d\boldsymbol{l}) &= \oint \boldsymbol{u}\cdot d\boldsymbol{u} \\ &= \oint (u\,du + v\,dv + w\,dw) \\ &= \frac{1}{2}\oint d(u^2+v^2+w^2) \\ &= 0 \end{aligned} \tag{2.33}$$

Which it remained to show; the proof is thus finished.

Since Γ is also given by a surface integral of the vorticity according to Eq. (**??**), wer notice that the flux of vorticity through a material loop in a nonviscous fluid is conserved. Changes of area of small material loops thus imply vortex stretching.

We will return later to a class of flow problems of considerable technical as well as everyday importance which can be described by Euler's equation, even if the fluid is viscous. The Kelvin-Helmholz theorem therefore has a large area of application. In this course we will use the theorem for the description of lift.

(To be cont'd)

2.4 Energy and momentum flow density for a compressible fluid

Any volume element in a fluid contains internal energy, and also kinetic energy and momentum if the flow velocity is different from zero. When a fluid is flowing, there is thus a transport of energy and momentum associated with the mass transport. It can be described by the *energy flow density* and the *momentum flow density*, the energy vs. the momentum per unit area per unit time. In these lecture notes the concept of flow density is essential for the description of energy dissipation, see below. It is also important for the understandingof energy transfer in turbulent flows.

We will find that for an ideal fluid the energy flow density is a vector parallel to the velocity vector, while for the definition of momentum flow density a *tensor* has to be used for the description. In both cases the derivation starts from the local time derivative of the energy or momentum per unit volume, and insertion for $\partial_t\boldsymbol{u}$ from the Euler equation (**??**). The resulting expressions are transformed into divergences of a vector or tensor by applying the continuity equation (**??**). By an ensuing spatial integration of the divergences, Gauss's theorem (**??**) transforms this into a surface integral of the vector or tensor. This vector or tensor may then be associated with a flow density.

In this special case of an ideal fluid we will concern ourselves about the part of the flow densities related to mass transport. In the next section we will see that for a viscous fluid there will be additional terms in the flow densities due to internal friction.

The energy flow density:

Let ϵ denote *internal energy per unit mass*. Neglecting contributions from potential energy, the variation with time of the energy per unit volume at a given point in space is

$$\partial_t(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u} + \rho\epsilon) \tag{2.34}$$

Transform the kinetic energy term by using the equation of motion and the continuity condition:[4]

$$\begin{aligned}
\partial_t(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}) &= \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\partial_t\rho + \rho\boldsymbol{u}\cdot\partial_t\boldsymbol{u} \\
&= -\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{div}(\rho\boldsymbol{u}) - \boldsymbol{u}\cdot\mathrm{grad}\,p - \rho\boldsymbol{u}\cdot((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) \\
&= -\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{div}(\rho\boldsymbol{u}) - \boldsymbol{u}\cdot\mathrm{grad}\,p - \frac{1}{2}\rho(\boldsymbol{u}\cdot\nabla)(\boldsymbol{u}\cdot\boldsymbol{u})
\end{aligned} \tag{2.35}$$

The gradient term can be rewritten using the thermodynamical relation (**??**) from Appendix C, og we obtain then an expression containing the *entropy* s and the *enthalpy* h *per unit mass*:

$$\partial_t(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}) = -\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{div}(\rho\boldsymbol{u}) - \rho\boldsymbol{u}\cdot\mathrm{grad}(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + h) + \rho T\boldsymbol{u}\cdot\mathrm{grad}\,s \tag{2.36}$$

For the transformation of the time derivative of the internal energy we apply the relations (**??**), (**??**) and (**??**)[5] from Appendix C, and also the continuity equation:

$$\begin{aligned}
\partial_t(\rho\epsilon) &= \epsilon\partial_t\rho + \rho\partial_t\epsilon \\
&= h\partial_t\rho + \rho T\partial_t s \\
&= -h\,\mathrm{div}(\rho\boldsymbol{u}) - \rho T\boldsymbol{u}\cdot\mathrm{grad}\,s
\end{aligned} \tag{2.37}$$

Accordingly:

$$\begin{aligned}
\partial_t(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u} + \rho\epsilon) &= -(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + h)\,\mathrm{div}(\rho\boldsymbol{u}) - \rho(\boldsymbol{u}\cdot\nabla)(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + h) \\
&= -\mathrm{div}\{\rho\boldsymbol{u}(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + h)\}
\end{aligned} \tag{2.38}$$

By integration and use of Gauss's theorem:

$$\begin{aligned}
\partial_t\int_V(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u} + \rho\epsilon)\,dV &= -\int_V \mathrm{div}\{\rho\boldsymbol{u}(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + h)\}\,dV \\
&= -\oint_S \rho\boldsymbol{u}(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + h)\cdot d\boldsymbol{S}
\end{aligned} \tag{2.39}$$

This shows that the vector

$$\boldsymbol{j}_E = \rho\boldsymbol{u}(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + h) \qquad \text{(compressible fluid)} \tag{2.40}$$

can be interpreted as an energy flow density, since the surface integral must equal the energy transported out of the volume per unit time.

For an *incompressible* fluid, Eq. (2.37) implies that $\partial_t(\rho\epsilon) = 0$, since $s =$ constant. With $h = \epsilon + p/\rho$ and $\epsilon =$ constant we then get the energy flow density for the incompressible case, with an arbitrary contribution from the internal energy suppressed in the notation:

$$\boldsymbol{j}_E = \rho\boldsymbol{u}(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + \frac{p}{\rho}) \qquad \text{(incompressible fluid)} \tag{2.41}$$

[4]In the case of an *incompressible* fluid we may then use the continuity equation $\mathrm{div}\,\boldsymbol{u} = 0$ to immediately transforming the RHS of eq. (2.35) into a total divergence, so that the result in Eq. (2.41) emerges directly via Gauss's theorem. See Problem 2.5.

[5]We assume that the process is adiabatic.

The momentum flow density:

The momentum per unit volume is $\rho\boldsymbol{u}$. Euler's equation and the requirement of continuity now gives

$$\begin{aligned}\partial_t(\rho\boldsymbol{u}) &= \rho\partial_t\boldsymbol{u} + \boldsymbol{u}\partial_t\rho \\ &= -\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \text{grad }\ p - \boldsymbol{u}\,\text{div}(\rho\boldsymbol{u})\end{aligned} \tag{2.42}$$

It will be useful to write this on component form:

$$\begin{aligned}\partial_t(\rho u_i) &= -\rho u_k\partial_k u_i - \partial_i p - u_i\partial_k(\rho u_k) \\ &= -\partial_i p - \partial_k(\rho u_i u_k) \\ &= -\partial_k\Pi_{ik}\end{aligned} \tag{2.43}$$

The quantity

$$\Pi_{ik} = p\delta_{ik} + \rho u_i u_k \tag{2.44}$$

is called the*momentum flow density tensor*. It is symmetric in the interchange of the indices i and k. Following a volume integration, Gauss's theorem now gives:

$$\begin{aligned}\partial_t\int_V \rho u_i\,dV &= -\int_V \partial_k\Pi_{ik}\,dV \\ &= -\oint_S \Pi_{ik}\hat{n}_k\,dS\end{aligned} \tag{2.45}$$

Here, $\hat{n}_k$ is a component of the unit normal vector in a point on the surface, so that $d\boldsymbol{S} = \hat{\boldsymbol{n}}\,dS$. The integrand $\Pi_{ik}\hat{n}_k$ is the flow of momentum component no. i per surface unit. After a multiplication of both sides by $\hat{\boldsymbol{e}}_i$ to find the variation with time of the total momemtum vector, the integrand on the RHS becomes

$$\Pi_{ik}\hat{\boldsymbol{e}}_i\hat{n}_k = p\hat{\boldsymbol{n}} + \rho\boldsymbol{u}(\boldsymbol{u}\cdot\hat{\boldsymbol{n}}) \tag{2.46}$$

Since the momentum itself is a vector parallel to $\boldsymbol{u}$, we cannot avoid a simultaneous specification of *two* directions—for both the velocity and the surface normal. We notice that the flow density of momentum through a surface has one term due to the pressure, and another which is dependent on the angle between $\boldsymbol{u}$ and $\hat{\boldsymbol{n}}$:

- $\boldsymbol{u} \parallel \hat{\boldsymbol{n}}$: $p + \rho\boldsymbol{u}\cdot\boldsymbol{u}$ in the $\hat{\boldsymbol{n}}$ direction
- $\boldsymbol{u} \perp \hat{\boldsymbol{n}}$: p in the $\hat{\boldsymbol{n}}$ direction

The momentum flow density tensor and the equation of motion:

We notice that the Euler equation can be written on the form

$$\rho\,D_t u_i = \partial_k\sigma_{ik} + \rho g_i \tag{2.47}$$

where the symmetrical *stress tensor* is given by the momentum density flow tensor as

$$\Pi_{ik} = -\sigma_{ik} + \rho u_i u_k \tag{2.48}$$

Here σ_{ik} contains only the isotropical pressure term$-p\delta_{ik}$. However, by generalizing the stress tensor to include also viscous stresses, we notice that the equation of motion for a viscous fluid follows from Eq. (2.47). This has been used in Appendix B in the derivation of the Navier-Stokes equation.

2.5 The impulse-momentum principle:

Eq. (2.43) is an alternative form of the Euler equation for an ideal compressible fluid without external field forces. It follows from the derivation and Eq. (??) that the corresponding expression for a viscous compressible fluid with field force (gravitational force) included, will emerge when τ_{ij} is included in Π_{ij} and $\rho \boldsymbol{g}$ is added on the RHS of (2.43). Eq. (2.46) will then be replaced by

$$\Pi_{ik}\hat{\boldsymbol{e}}_i\hat{n}_k = p\hat{\boldsymbol{n}} + \rho\boldsymbol{u}(\boldsymbol{u}\cdot\hat{\boldsymbol{n}}) - \tau_{ik}\hat{\boldsymbol{e}}_i\hat{n}_k \tag{2.49}$$

where, for $\nabla\cdot\boldsymbol{u} = 0$,

$$\tau_{ik}\hat{\boldsymbol{e}}_i\hat{n}_k = \mu(\hat{n}_k\nabla u_k + (\hat{\boldsymbol{n}}\cdot\nabla)\boldsymbol{u}) \tag{2.50}$$

Conservation of momentum in the flow is then expressed by the generalized form of Eq. (2.45),

- *the general impulse-momentum principle for an incompressible fluid*:

$$\partial_t \int_V \rho\boldsymbol{u}\, dV + \int_S \rho\boldsymbol{u}(\boldsymbol{u}\hat{\boldsymbol{n}})\, dS = -\int_S p\hat{\boldsymbol{n}}\, dS + \mu\int_S \hat{n}_k\nabla u_k\, dS + \mu\int_S (\hat{\boldsymbol{n}}\nabla)\boldsymbol{u}\, dS + \boldsymbol{g}\int_V \rho\, dV \tag{2.51}$$

It applies to a stationary and nondeformable control volume V with enclosing surface S.

For $\mu = 0$ and a stationary flow we recognize the most usual expression for technical use:

$$\int_S \rho\boldsymbol{u}(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})\, dS = \Sigma\boldsymbol{F} \tag{2.52}$$

The normal vector $\hat{\boldsymbol{n}}$ points out of the volume; the minus sign on the pressure term thus denotes a force contribution *from* the surroundings *on* the fluid in the control volume. If a part of the surface consists of the wall of an impermeable conduit[6], the integral of pressure over this part will give the constraint forces.

2.6 Energy dissipation in a viscous incompressible fluid

The viscosity of a fluid makes mechanical energy getting lost as heat. In contrast to the idealized result from a previous section, there must then be a difference between the time derivative of the energy in a volume and the net energy flow rate through the surface of the volume; this will be the energy loss.

We will derive this *dissipative* term for the case of an adiabatic flow of an incompressible fluid. With both the entropy s and the density ρ constant, also the internal energy ϵ will be a constant, according to Eq. (2.37). konstant. The time derivative

$$E_{kin} = \frac{1}{2}\rho\int_V \boldsymbol{u}\cdot\boldsymbol{u}\, dV \tag{2.53}$$

will now be of interest.

After taking the time derivative of Eq. (2.53) we insert for $\partial_t\boldsymbol{u}$ from the Navier-Stokes equation, for our purposes written on the primitive form (??) with the viscous tensor defined as in Eq. (??):

$$\partial_t u_i = -u_k\partial_k u_i - \frac{1}{\rho}\partial_i p + \frac{1}{\rho}\partial_k\tau_{ik} \tag{2.54}$$

[6] Physical terminology for *pipe wall*.

(For simplicity of notation we neglect the gravity term.) We know from the section about energy and momentum flow density that after insertion in the expression for the time derivative of E_{kin}, the first two terms on the RHS can be reduced to the divergence of the vector $\boldsymbol{j}_E$ from Eq. (2.41); this vector represents the energy flow density associated with the mass transport. The remaining term on the RHS of (2.54) has the potential to give rise to new effects. We rewrite it to get:

$$u_i \partial_k \tau_{ik} = \partial_k (u_i \tau_{ik}) - \tau_{ik} \partial_k u_i \tag{2.55}$$

The first term on the RHS is also a pure divergence. We have then identified a new contribution to the $\boldsymbol{j}_E^{visc}$ to the energy flow density, which represents the momentum flux associated with inner friction:

$$(\boldsymbol{j}_E^{visc})_i = -u_k \tau_{ik} \qquad \text{(incompressible fluid)} \tag{2.56}$$

Remember that τ_{ik} must appear as an extra subtractive term in the momentum flow density tensor Π_{ik} in Eq (2.48). Since a momentum flow will always be associated with an energy flow, the interpretation of $\boldsymbol{j}_E^{visc}$ as a viscous contribution to the energy flow density follows immediately.[7]

We now let the volume V in Eq. (2.53) be large enough that it encompasses the whole fluid with enclosing, so that no energy flow contributions from the surface appear in the time derivative of E_{kin}. Then:

$$\begin{aligned} \partial_t E_{kin} &= \rho \int_V u_i \partial_t u_i \, dV \\ &= -\int_V \tau_{ik} \partial_k u_i \, dV \end{aligned} \tag{2.57}$$

For an incompressible fluid we know from Eq. (**??**) that

$$\tau_{ik} = \mu \left(\partial_i u_k + \partial_k u_i \right) \tag{2.58}$$

The symmetry property makes possible a reformulation of the integrand:

$$\begin{aligned} \tau_{ik} \partial_k u_i &= \mu \left(\partial_i u_k + \partial_k u_i \right) \partial_k u_i \\ &= \frac{1}{2} \mu \left(\partial_i u_k + \partial_k u_i \right)^2 \end{aligned} \tag{2.59}$$

Whe have thus found an expression for *the energy loss rate for a viscous incompressible fluid*:

$$\partial_t E_{kin} = -\frac{1}{2} \mu \int_V \left(\partial_i u_k + \partial_k u_i \right)^2 dV \tag{2.60}$$

It is consistent that for a positive μ and a quadratic integrand, this equation implies that the mechanical energy will always be lost as heat with time.

(to be cont'd)

[7] We got this explicit demonstration as a bonus since we introduced the Navier-Stokes equation on the form (**??**)/(**??**) instead of on the final form (**??**).