Chapter 4

Viscous flow: Stokes's law

In this chapter we will treat some characteristics of viscous flow. It will be done by treating in detail a case of creeping flow.

4.1 Simple solutions of the Navier-Stokes equation

The nonlinear term in the Navier-Stokes equation causes that exact analytical solutions can only be found in a minority of the physically relevant situations. It is characteristical for the two most wellknown solutions for a non-negligible viscous term,

- stationary laminar pipe flow (Poiseuille flow)
- stationary laminar channel flow

both with a constant flow cross section, that the nonlinear convective acceleration term does not play any role at all. However, no assumption about creeping flow is involved—the nonlinear term is zero for geometrical reasons, and the solutions are in principle valid also for $\mathcal{R} \gg 1$. In practice, as is well known, the validity is limited by upper critical values for $\mathcal{R}$ of order 10^3, where the laminar flow becomes unstable.

Another type of flow where the nonlinear term is not zero, but plays a secondary role, is

- stationary laminar rotating Couette flow

i.e., flow in the annulus between two rotating coaxial cylinders. In the absence of instabilities (which may result in a completely different flow pattern), an exact solution can easily be found by assume a rotational symmetry around the axis as well as translational symmetry along the axis.

We will assume that the method of solution for the first two problems is known from lectures or training problems in the elementary fluid mechanics course. (In the elementary course they are often treated in a simplified way, without direct reference to the Navier-Stokes equation; a proper treatment can be found for instance in [Palm 1977] and [Ytrehus 1981].) We will return to the third in a problem in this chapter.

Some other exact solutions in the nonlinear case, of varying degree of physical relevance, have been derived in [Landau and Lifshitz 1987]. The perhaps most interesting one is von Kármán's solution for the flow around an infinite plate rotating in a viscous fluid around an axis perpendicular to the plate. It is in principle valid for all values of the parameters. However,

Figure 4.1: Viscous flow past a sphere, in a plane through the axis of flow, for arbitrary ϕ

also here the validity ought in practice to be limited by a critical value of a suitably defined Reynolds number, which assumedly would involve the angular frequency, the kinematical viscosity and the local radius from the axis of rotation.

4.2 Creeping flow past a sphere

Creeping flow, where the nonlinear term is assumed to be zero because $\mathcal{R} \ll 1$, presents an opportunity to use wellknown techniques for the solution of partial differential equations. Such conditions arise in various geometries.[1] We will consider a stationary incompressible viscous fluid flow past a sphere at rest, for $\mathcal{R} \ll 1$. This is a case where the Navier-Stokes equation is linearized due to the assumption about creeping flow, but where the method of solution is nontrivial in spite of the simple geometry. This solution for the flow problem is the basis for calculation of the drag force on a sphere moving through a viscous fluid (Stokes's law).

4.2.1 Solution for velocity and pressure

Geometry and boundary conditions:

Figure 4.1 shows the geometry. The velocity vector for the homogeneous flow at large distances from the sphere is $\boldsymbol{u}_0$. The sphere radius is R. We will neglect the force of gravity. The problem is well suited for the introduction of spherical coordinates (see Appendix A). The problem is simplified by the rotational symmetry around the axis through the sphere's center in the $\boldsymbol{u}_0$ direction, so that we do not need to consider the ϕ coordinate. Additionally, the assumption about creeping flow makes the streamline pattern mirror symmetric about a plane through the

[1] A type we will not treat in detail here, arise in lubricated bearings due to small linear dimensions in the gap between axle and bearing (cf. the *hydrodynamical theory of lubrication*, [Sommerfeld 1964]). Creeping flow is thus of considerable practical relevance—think about it next time you start the motor of your car.

center of the sphere perpendicular to the flow axis. With the velocity vector decomposed into a radial component u_r and a tangential component u_θ as in the figure, we get the geometrical relations (with $u_0 = |\boldsymbol{u}_0|$ and $\hat{\boldsymbol{n}} = \boldsymbol{r}/|\boldsymbol{r}|$):

$$\begin{aligned} \boldsymbol{u}_0 \cdot \hat{\boldsymbol{n}} &= u_0 \cos\theta && (4.1)\\ (\boldsymbol{u}_0)_r &= u_0 \cos\theta && (4.2)\\ (\boldsymbol{u}_0)_\theta &= -u_0 \sin\theta && (4.3) \end{aligned}$$

With $r = |\boldsymbol{r}|$ the boundary conditions become:

$$\begin{aligned} u_r &= u_\theta = 0 && (r = R) && (4.4)\\ u_r &\rightarrow u_0 \cos\theta && (r \rightarrow \infty) && (4.5)\\ u_\theta &\rightarrow -u_0 \sin\theta && (r \rightarrow \infty) && (4.6) \end{aligned}$$

Equations of motion and continuity, formulation of the method of solution:

We will find the fluid velocity $\boldsymbol{u}$ and the pressure p by simultaneous solution of the equation of motion (**??**) and the equation of continuity (**??**):

$$\begin{aligned} \nabla p &= \mu \nabla^2 \boldsymbol{u} && (4.7)\\ \nabla \cdot \boldsymbol{u} &= 0 && (4.8) \end{aligned}$$

In the following we will use the same method of solution as [Landau and Lifshitz 1987].[2] We first employ the curl operator on both sides of the equation of motion. Using Eq. (**??**) we then get the equation of motion on a form which does not involve the pressure. With the order of derivations changed:[3]

$$\nabla^2 \operatorname{curl} \boldsymbol{u} = 0 \qquad (4.9)$$

Subsequently, the continuity condition is transformed. We introduce

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}' + \boldsymbol{u}_0 && && (4.10)\\ \lim_{r \rightarrow \infty} \boldsymbol{u}' &= 0 && \text{(boundary condition)} && (4.11) \end{aligned}$$

Because $\boldsymbol{u}_0 =$ konstant we get $\operatorname{div} \boldsymbol{u}' = 0$, and according to Eq. (**??**) we can then express $\boldsymbol{u}'$ by a *vector potential* $\boldsymbol{A}$:

$$\boldsymbol{u} = \operatorname{curl} \boldsymbol{A} + \boldsymbol{u}_0 \qquad (4.12)$$

To determine the properties of the vector $\boldsymbol{A}$, we make use of the *parity properties* of the vectors, which are treated in Appendix D. Because $\boldsymbol{u}$ is a *polar* vector, also the RHS of Eq. (4.12) must be a purely polar quantity. For the vector product of the polar ∇ and $\boldsymbol{A}$ to be polar, $\boldsymbol{A}$ has to be an *axial* vector. Since $\boldsymbol{u}' = \boldsymbol{u}'(\boldsymbol{u}_0, \boldsymbol{r})$, the functional dependence $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{u}_0, \boldsymbol{r})$ follows. Because the equation of motion for creeping flow is linear, $\boldsymbol{A}$ has to depend linearly on $\boldsymbol{u}_0$. Since we consider flow around a spherically symmetric body, the vector product $\boldsymbol{r} \times \boldsymbol{u}_0$

[2] The seemingly simpler method in [Sommerfeld 1964] depends on concealed details related to a decomposition into *spherical harmonics*, a type of functions related to *Legendre polynomials*. See [Papatzacos 2003] for an example of the use of Legendre polynomials for the solution of the Laplace equation.

[3] Notice that $\operatorname{curl} \boldsymbol{u} \neq 0$—we have left the potential flow formalism, because the viscous term has been included in the equation of motion.

involving polar vectors is the only axial combination which may enter $\boldsymbol{A}$. With $f'(r)$ a *scalar* function, we thus have

$$\boldsymbol{A} = f'(r)\,\hat{\boldsymbol{n}}\times\boldsymbol{u}_0 \tag{4.13}$$

However, since $f'(r)$ has no variation in the angular direction, we may introduce another scalar function $f(r)$ which has $f'(r)$ as its first derivative:

$$f'(r)\,\hat{\boldsymbol{n}} = \nabla f(r) \tag{4.14}$$

Thus we can write

$$\boldsymbol{u} = \mathrm{curl}\,(\nabla f\times\boldsymbol{u}_0) + \boldsymbol{u}_0 \tag{4.15}$$

Since $\boldsymbol{u}_0 = \text{constant}$, we find

$$\begin{aligned}(\nabla f\times\boldsymbol{u}_0)_i &= \epsilon_{ijk}\partial_j f u_{0k}\\ &= \epsilon_{ijk}\partial_j(f\boldsymbol{u}_0)_k\\ &= (\mathrm{curl}\,(f\boldsymbol{u}_0))_i\end{aligned} \tag{4.16}$$

and thus explicitly

$$\boldsymbol{u} = \mathrm{curl}\,\mathrm{curl}\,(f\boldsymbol{u}_0) + \boldsymbol{u}_0 \tag{4.17}$$

Eq. (4.15) / (4.17) is essentially a transformated form of the continuity condition, with parity considerations added. All that remains now is to solve this equation together with the equation of motion (4.9), and thus obtain $f(r)$.

Use of the method of solution for simplification:

Inserting (4.15) into (4.9) and using Eqs. (**??**) and (**??**), as well as using that $\boldsymbol{u}_0$ is a constant, is the first step:

$$\begin{aligned}\nabla^2\,\mathrm{curl}\,\mathrm{curl}\,(\nabla f\times\boldsymbol{u}_0) &= \nabla^2\,(\mathrm{grad}\,\mathrm{div} - \nabla^2)\,(\nabla f\times\boldsymbol{u}_0)\\ &= -(\nabla^2)^2\,(\nabla f\times\boldsymbol{u}_0)\\ &= -((\nabla^2)^2\,\mathrm{grad}\,f)\times\boldsymbol{u}_0\\ &= 0\end{aligned} \tag{4.18}$$

The last equal sign in this equation embodies the intersection of information from the equations of motion and continuity and the parity conditions. Since $\mathrm{grad}\,f \propto \hat{\boldsymbol{n}}$, a vector which may point in an arbitrary direction, this means that

$$(\nabla^2)^2\,\mathrm{grad}\,f = 0 \tag{4.19}$$

Changing the order of derivations and then integrating once we get

$$(\nabla^2)^2\,f(r) = \text{constant} \tag{4.20}$$

However, the constant must equal zero according to Eq. (4.11) (the boundary condition at $r\to\infty$), since $\boldsymbol{u}'$ essentially is a second derivative of f. Using the expression (**??**) for the Laplace operator applied to a scalar, we get

$$\begin{aligned}(\nabla^2)^2\,f &= \frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr})\,\nabla^2 f \;=\; 0\\ r^2\frac{d}{dr}\nabla^2 f &= -2a\\ \nabla^2 f &= \frac{2a}{r} + A\end{aligned} \tag{4.21}$$

where a and A are constants of integration. Using once more the boundary condition at $r \to \infty$, we realize that $A = 0$. Further integrations give[4]

$$\begin{aligned} \frac{1}{r^2}\frac{d}{dr}(r^2\frac{df}{dr}) &= \frac{2a}{r} \\ r^2\frac{df}{dr} &= ar^2 - b \\ f(r) &= ar + \frac{b}{r} + c \end{aligned} \tag{4.22}$$

b and c are constants of integration; c will not concern us since $\boldsymbol{u}'$ only depends on the derivatives of $f(r)$, so we will skip it in what follows. Thus, after all the vector manipulations we are left with a simple result, which may now be inserted into Eq. (4.17) to obtain the velocity vector.

Final solutions:

Insertion of Eq. (4.22) into (4.17) entails the following calculation in a cartesian notation, where we apply the useful result (**??**) from Appendix H, as well as the contraction relation (**??**):

$$\begin{aligned} (\mathrm{curl}\,\mathrm{curl}\,(\boldsymbol{u}_0 r^N))_i &= \epsilon_{ijk}\partial_j\epsilon_{klm}\partial_l(u_{0m}r^N) \qquad (N \in \{-1,1\} \text{ her}) \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})u_{0m}\partial_j\partial_l r^N \\ &= u_{0m}\partial_m\partial_i r^N - u_{0i}\partial_j\partial_j r^N \\ &= N\{u_{0i}r^{N-2} + (N-2)r^{N-4}x_i x_m u_{0m} \\ &\quad - u_{0i}3r^{N-2} - (N-2)r^{N-4}u_{0i}x_m x_m\} \\ &= \frac{N}{r^{2-N}}\{u_{0i} + (N-2)\hat{n}_i(\boldsymbol{u}_0\cdot\hat{\boldsymbol{n}}) - 3u_{0i} - (N-2)u_{0i}\} \end{aligned}$$

Equivalently:

$$\mathrm{curl}\,\mathrm{curl}\,(\boldsymbol{u}_0 r^N) = \frac{N}{r^{2-N}}\{-N\boldsymbol{u}_0 + (N-2)\hat{\boldsymbol{n}}(\boldsymbol{u}_0\cdot\hat{\boldsymbol{n}})\} \tag{4.23}$$

This result expresses the velocity by the two integration constants a and b:

$$\boldsymbol{u} = \boldsymbol{u}_0 - a\,\frac{\boldsymbol{u}_0 + \hat{\boldsymbol{n}}(\boldsymbol{u}_0\cdot\hat{\boldsymbol{n}})}{r} + b\,\frac{-\boldsymbol{u}_0 + 3\hat{\boldsymbol{n}}(\boldsymbol{u}_0\cdot\hat{\boldsymbol{n}})}{r^3} \tag{4.24}$$

The integration constants are eventually determined from the boundary condition (4.1) at the sphere surface ($r = R$). Because $\hat{\boldsymbol{n}}$ has an arbitrary direction, the coefficients of $\boldsymbol{u}_0$ and $\hat{\boldsymbol{n}}(\boldsymbol{u}_0\cdot\hat{\boldsymbol{n}})$ must both equal zero:

$$\begin{aligned} 1 - \frac{a}{R} - \frac{b}{R^3} &= 0 \\ -\frac{a}{R} + \frac{3b}{R^3} &= 0 \end{aligned}$$

Which gives

$$a = \frac{3}{4}R \tag{4.25}$$

$$b = \frac{1}{4}R^3 \tag{4.26}$$

[4] Hard work ...

The flow velocity as a function of $\boldsymbol{r}$ has thus been found.

Subsequently we get the pressure by inserting Eq. (4.15) into (4.7), interchanging the order of derivations, and using Eq. (**??**):

$$\begin{aligned}
\operatorname{grad} p &= \mu \operatorname{curl}(\nabla(\nabla^2 f)\times \boldsymbol{u}_0) \\
&= \mu\, \epsilon_{ijk}\hat{\boldsymbol{e}}_i\partial_j\epsilon_{kmn}\partial_m(\nabla^2 f)u_{0n} \\
&= \mu\,(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})\,\partial_j\partial_m(\nabla^2 f)u_{0n}\hat{\boldsymbol{e}}_i \\
&= \operatorname{grad}\{\mu\, \boldsymbol{u}_0\cdot\operatorname{grad}(\nabla^2 f)\} - \mu\, \boldsymbol{u}_0\,(\nabla^2)^2 f
\end{aligned} \tag{4.27}$$

From the considerations in connection with Eq. (4.20) we know that the second term on the RHS is zero. It follows that we may write

$$\begin{aligned}
p &= p_0 + \mu\, \boldsymbol{u}_0\cdot\operatorname{grad}(\nabla^2 f) \\
&= p_0 + \mu\, \boldsymbol{u}_0\cdot\operatorname{grad}\frac{2a}{r} \\
&= p_0 + 2a\mu\, u_{0i}\,\frac{-x_i}{r^3} \\
&= p_0 - 2a\mu\,\frac{\boldsymbol{u}_0\cdot\hat{\boldsymbol{n}}}{r^2}
\end{aligned} \tag{4.28}$$

where p_0 is a constant. To obtain this expression we first used Eq. (4.21) and then (**??**).

Using Eqs. (4.24), (4.25), (4.26) and (4.28) and the geometrical relations (4.1), (4.2) and (4.3), we may express the final set of solutions for velocity and pressure as:[5]

$$\begin{aligned}
u_r &= u_0\cos\theta\,\{1-\frac{2a}{r}+\frac{2b}{r^3}\} \\
&= u_0\cos\theta\,\{1-\frac{3}{2}\frac{R}{r}+\frac{1}{2}(\frac{R}{r})^3\}
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
u_\theta &= -u_0\sin\theta\,\{1-\frac{a}{r}-\frac{b}{r^3}\} \\
&= -u_0\sin\theta\,\{1-\frac{3}{4}\frac{R}{r}-\frac{1}{4}(\frac{R}{r})^3\}
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
p &= p_0 - 2a\mu u_0\,\frac{\cos\theta}{r^2} \\
&= p_0 - \frac{3}{2}\frac{\mu u_0 R}{r^2}\cos\theta
\end{aligned} \tag{4.31}$$

4.2.2 The drag force: Stokes's law

In a calculation of the resultant force acting on the sphere moving through the fluid, we must consider

- the thermodynamical pressure p
- the viscous shear stress $\tau_{\theta r}$
- the viscous normal stress τ_{rr}

[5] In [Tritton 1988] it was introduced without a derivation.

at the surface of the sphere. We must sum over the stress components at all surface elements and integrate over the surface. The total force F on the sphere in the $\boldsymbol{u}_0$ direction is

$$F = \oint_S (-p\cos\theta + \tau_{rr}\cos\theta - \tau_{\theta r}\sin\theta)\, dS \tag{4.32}$$

The expressions (**??**) and (**??**) for the stresses can be inserted here. We might also construct these expressions at $r = R$ by considering local Cartesian coordinate systems in every point on the surface, with one axis pointing radially; then τ_{rr} is obtained from Eq. (**??**) and $\tau_{\theta r}$ from the basic expression used in elementary courses in the definition of viscosity. Anyway, for $r = R$ a simple result emerges:

$$\tau_{rr} = 2\mu \frac{\partial u_r}{\partial r}|_{r=R} = 0 \tag{4.33}$$

$$\begin{aligned} \tau_{\theta r} &= \mu \frac{\partial u_\theta}{\partial r}|_{r=R} \\ &= -\mu u_0 \sin\theta \{\frac{a}{R^2} + 3\frac{b}{R^4}\} = -\frac{3}{2}\frac{\mu u_0}{R}\sin\theta \end{aligned} \tag{4.34}$$

$$\begin{aligned} -p\cos\theta - \tau_{\theta r}\sin\theta &= -p_0\cos\theta + \frac{3}{2}\frac{\mu u_0}{R}\cos^2\theta + \frac{3}{2}\frac{\mu u_0}{R}\sin^2\theta \\ &= -p_0\cos\theta + \frac{3}{2}\frac{\mu u_0}{R} \end{aligned} \tag{4.35}$$

This means that apart from the constant p_0 term whose contribution integrated over the sphere surface will cancel, all points on the surface contribute equally to the drag force:

$$\begin{aligned} F &= \frac{3}{2}\frac{\mu u_0}{R}\oint_S dS \\ &= \frac{3}{2}\frac{\mu u_0}{R} 4\pi R^2 \end{aligned} \tag{4.36}$$

Or:

$$\boldsymbol{F} = 6\pi\mu R\, \boldsymbol{u}_0 \tag{4.37}$$

This experssion is known as *Stokes's law*. With a drag coefficient C_D introduced in a standard way

$$F = \frac{1}{2}C_D\rho u_0^2 D^2 \qquad (D = 2R) \tag{4.38}$$

we get

$$C_D = \frac{6\pi}{\mathcal{R}} \qquad (\mathcal{R} = \frac{u_0 D}{\nu}) \tag{4.39}$$

Figure 4.2 shows a comparison of the prediction from Stokes's law with experimental results. The theoretical result is confirmed (within drawing accuracy) in the limit $\mathcal{R} \to 0$. The agreement is less good for $\mathcal{R} > 0.5$.

We have considered a problem with a simple geometry, and the wellknown end result (4.37) has a very simple form. We notice that even such simple conditions, together with a linearized version of the Navier-Stokes equation, may involve quite acrobatic mathematics before the end

Figure 4.2: Drag force on a sphere at low Reynolds numbers. The solid line corresponds to Stokes's law

result emerges.[6] However, we have obtained a confirmation that the assumptions involved in the description of creeping flow are valid, and the result is an incentive to study related applications.

4.2.3 Limitations of validity

However, simplifications may not be as simple as they look at first sight. Examination of the solutions for velocity and pressure for creeping flow past a sphere, Eqs. (4.29), (4.30) and (4.31), will show that the creeping flow approximation itself must lose validity for large enough r, for all finite values of $\mathcal{R}$. For $r \gg R$ one gets for the order of magnitude of certain terms in the Navier-Stokes equation,

$$|\,(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\,| \sim \frac{u_0^2 R}{r^2} \qquad , \qquad |\,\frac{1}{\rho}\nabla p\,| \sim \frac{\mu u_0 R}{\rho r^3} \tag{4.40}$$

[6]The actual *functional form* in Eq. (4.37) follows from Eq. (4.38) by dimensional analysis, if it is presupposed that C_D must depend on such a power of $\mathcal{R}$ that $\boldsymbol{F}$ is linear in all the parameters. This presupposition is consistent with the general assumptions for creeping flow. However, to obtain the particular multiplicative numerical constant, a mathematical apparatus with a certain complexity had to be mobilized.

so that the assumption about a negligible nonlinear term is inconsistent for large r:

$$\frac{|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}|}{|\frac{1}{\rho}\nabla p|} \sim \frac{u_0 r}{\nu} = \frac{r}{2R}\mathcal{R} \tag{4.41}$$

Stated alternatively: The Stokes solutions for velocity and pressure are are only valid for regions where RE expressed by local radius r instead of D are less than unity.

In 1910 the Swede Oseen introduced a correction for this deficiency by keeping a linearized version of the convective acceleration term in the Navier-Stokes equation:

$$\rho(\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u} = -\nabla p + \mu\nabla^2\boldsymbol{u} \tag{4.42}$$

In the limit $r \to \infty$ this must evidently be a good approximation. The modified set of solutions for $\boldsymbol{u}$ and p gives rise to the following lowest order modification of Stokes's formula:

$$F = 6\pi\mu R u_0\{1 + \frac{3}{16}\mathcal{R}\} \tag{4.43}$$

Figure 4.2 shows that the sign of the correction term is correct. However, the condition $\mathcal{R} \ll 1$ must still be imposed: For $\mathcal{R} = 5$, Oseen's correction term is roughly the double of what it ought to be to get agreement with the experimental results.[7]

The corresponding problem for creeping flow in *two* dimensions, past a cylinder, has no solution unless one introduces Oseen's term in the equation of motion. If not, one will not find a solution for the velocity which satisfies the boundary conditions at the cylinder surface while also predicting an unperturbed uniform flow for $r \to \infty$ (*Stokes's paradox*, see [Birkhoff 1960]).

The boundary layer for viscous flow extends to infinity, qualitatively speaking, based on the slow variations in r we have found in this chapter. More practically: If a viscosimeter

[7]A commentary about the use of Eqs. (4.37) and (4.43) i *separator theory* (a topic in chemical chemical process theory) is in order. If the sphere had been a spherical fluid bubble with viscosity μ', with another fluid with viscosity μ flowing past it, that would result in an extra multiplicative correction factor $(\mu' + \frac{2}{3}\mu)/(\mu' + \mu)$ in Eq. (4.37) [Landau and Lifshitz 1987]. By equating the drag force to the difference between weight and buoyancy, one gets the following expression for the fall (or rise) velocity:

$$u_0 = \frac{1}{6}\frac{\mu' + \mu}{3\mu' + 2\mu}\frac{D^2 g}{\mu}(\rho - \rho') \qquad (\mathcal{R} \ll 1)$$

This formula is commonly used for water bubbles in oil, and for gas bubbles in oil or water [Asheim 1985].

For fluid bubbles in a gas one usually has $\mu' \gg \mu$, so that the correction factor mentioned above can be omitted. Oseen's correction term in Eq. (4.43) would give $C_D = 9\pi/8$ for $\mathcal{R} \gg 1$, which is of the correct *form* for *turbulent* drag. In this limit—outside the equation's range of validity!—Oseen's term may be tentatively used to estimate the fall velocity for bubbles in a turbulent flow. Balancing the forces we get:

$$u_0 = k_s\sqrt{\frac{\rho' - \rho}{\rho}} \quad , \quad k_s^2 = \frac{16}{27}gR \qquad (\mathcal{R} \gg 1)$$

In practice a value of k_s assumedly obtained from fitting to experimental results is used (see [Asheim 1985], [Lydersen 1979]), which is a factor 3.20 larger. (Close to π, but a numerical coincidence.)

We notice that Oseen's formula predicts the gradual transition from a linear to a quadratic law for the resistance due to friction. However, it was derived under the assumption of laminar flow, and it is not well suited for *quantitative* calculations for turbulent flow past spherical bubbles of fluid.

Finally it may be mentioned that if the *surface tension* contributes, for instance for large gas bubbles arising in a stationary fluid, one finds a different expression for u_0. See Problem 7.27 in [Franzini and Finnemore 2002].

based on Eq. (4.37) were to be constructed, the diameter of the observation tank would have to be at least 100× the sphere diameter, if less than a 2 % systematic deviation is required. (See also Problem 4.2.)

4.3 Problems

Problem 4.1 The annulus between two rotating coaxial cylinders is filled by a fluid with viscosity μ. The inner and outer cylinders, with radii R_1 and R_2, have rotational angular frequencies Ω_1 and Ω_2.

a) Find the velocity distribution in the fluid for a stationary laminar Couette flow.

b) Find the torque per unit length along the axis which is transferred through the fluid.

Problem 4.2 For a) nonviscous flow and b) Stokes flow past a sphere, determine the distance from the sphere perpendicular to the flow (measured in units of r/R) where the difference from the velocity's unperturbed value has fallen to 1 %. Use the expression in Problem 3.2 and also Eq. (4.30).

Problem 4.3 a) Find the largest possible diameter for water drops falling in air with a velocity where Stokes's law can be used in the calculation. *Given:* $\rho_{air} \approx 1.2\,\mathrm{kg\,m^{-3}}$, $\mu_{air} \approx 1.8\times10^{-5}\,\mathrm{kg\,m^{-1}\,s^{-1}}$.

b) Suppose that a $1\,\mathrm{m\,s^{-1}}$ wind generates a turbulence with intensity 10 % of the mean velocity. Determine how small a water drop in this wind has to be, for its fall velocity due to its weight to be negligible.