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PARAMETRIC REPRESENTATION OF UNIVALENT FUNCTIONS

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Let S be the class of all functions $w = f(z)$ which are holomorphic and univalent in the circle $E = \{z: |z| < 1\}$ and normalized by the conditions $f(0) = 0$, $f'(0) = 1$.

In this paper we shall solve the problem of parametrically representing S by establishing necessary and sufficient conditions that a function $w = f(z)$ belongs to S. In addition we indicate an application of our results to the solution of extremal problems in the class S and the class P of all functions $w = h(z)$, $h(0) = 1$, which are holomorphic in the circle E and have positive real part.

1. Let $\mathfrak{M}$ be the class of all nondecreasing functions $\mu(x, y)$ of two variables in the region $x \geq 0$, $-\pi \leq y \leq \pi$ which are normalized by the conditions $\mu(x, -\pi) = \mu(0, y) = 0$, $\mu(x, \pi) = x$.

It follows directly from the definition of the class $\mathfrak{M}$ that for each fixed y, $-\pi \leq y \leq \pi$, the function $\mu(x, y)$ is absolutely continuous in x and, consequently, the derivative $\mu'_x(x, y)$ exists for almost all x, $x > 0$, $\mu(x, y)$ is a measurable function of x for fixed y, a nondecreasing function of y, $-\pi \leq y \leq \pi$ for each fixed x, $x > 0$, and is normalized by the conditions $\mu'_x(x, -\pi) = 0$, $\mu'_x(x, \pi) = 1$.

We shall say that the sequence $\mu_n(x, y)$ $(n = 1, 2, 3, \cdots)$ of functions belonging to $\mathfrak{M}$ converges to the function $\mu(x, y) \in \mathfrak{M}$ if at each point of continuity of $\mu(x, y)$, $\lim_{n\to\infty} \mu_n = \mu$.

The class $\mathfrak{M}$ is dense in itself with respect to the above definition of the convergence of sequences in $\mathfrak{M}$.

By Φ we shall denote the set of all functions $f(z, x, y)$ which are continuous in the region $E \times [0, \infty) \times [-\pi, \pi]$, analytic with respect to z in the circle E and satisfy the condition $|f(z, x, y)| \leq e^{x} K(r)$, where $K(r)$ is a constant which depends only on $r = |z| < 1$.

Let $f(z, x, y)$ be any function in Φ and let μ_n be any sequence of functions in $\mathfrak{M}$ converging to the function $\mu(x, y) \in \mathfrak{M}$. Then:

1) the limit

$$\lim_{n\to\infty} \int_0^x \int_{-\pi}^{\pi} f(z, x, y)\, d\mu_n(x, y) = \int_0^x \int_{-\pi}^{\pi} f(z, x, y)\, d\mu(x, y)$$

exists uniformly with respect to x, $0 \leq x \leq A$ and $z \in E_r = \{z: |z| < r < 1\}$;

2) the Stieltjes integral

$$\int_0^\infty \int_{-\pi}^{\pi} f(z, x, y)\, d\mu(x, y), \quad \mu \in \mathfrak{M},$$

converges uniformly within E and equicontinuously with respect to the class $\mathfrak{M}$.

From this it follows immediately that the limit

$$\lim_{n\to\infty} \int_0^\infty \int_{-\pi}^{\pi} f(z, x, y)\, d\mu_n(x, y) = \int_0^\infty \int_{-\pi}^{\pi} f(z, x, y)\, d\mu(x, y)$$

exists uniformly within E.

2. Consider the differential equation

$$\frac{dw}{dx} = -w \int_{-\pi}^{\pi} g(w, y)\, d\mu_x'(x, y) \tag{1}$$

where $g(w, y) = (1 + e^{iy}w)/(1 - e^{iy}w)$ with the initial condition $w(x)|_{x=0} = z$, $z \in E$. Here the function $\mu(x, y) \in \mathfrak{M}$ and the integral in (1) is a Stieltjes integral.

We shall denote the solution of the differential equation in (1) which satisfies the initial condition by $f(z, x; \mu)$.

Theorem 1. *For the function* $w = f(z)$ *to belong to the class* S *it is necessary and sufficient that it can be represented in the form*

$$f(z) = \lim_{x\to\infty} e^x f(z, x; \mu), \quad \mu \in \mathfrak{M}. \tag{2}$$

We indicate the course of the proof of Theorem 1. Let $\mu(x, y)$ be an arbitrary function from the class $\mathfrak{M}$. We replace the differential equation in (1) and the initial condition by the integral equation

$$w = z \exp\left\{-\int_0^x \int_{-\pi}^{\pi} g(w, y)\, d\mu(x, y)\right\} \tag{3}$$

which is obtained by dividing through (1) by w and integrating with respect to x from 0 to x. By solving (3) by the method of successive approximations (see, e.g., [1], Russian pp. 96–97), we find that the solution $f(z, x; \mu)$ is regular in the circle E and continuous in the interval $0 < x < \infty$ and, in addition, $f(0, x; \mu) = 0$, $f_x'(0, x; \mu) = e^{-x}$.

It follows from the easily proven uniqueness of the solution of equation (1) that the function $f(z, x; \mu)$ is univalent in E for each value of x in the interval $[0, \infty)$. It remains to establish the existence of the uniform limit with respect to z in E in (2). To this end we place $f(z, x; \mu)$ in (1) and rewrite it in the form

$$[e^x f(z, x; \mu)]_x' = e^x f(z, x; \mu)\,[1 - g(f(z, x; \mu) y)], \tag{4}$$

noting that the function which appears on the right-hand side of equation (4) belongs to the class Φ.*

By integrating (4) with respect to x from 0 to x and taking the limit as x tends to infinity we find that the function $f(z)$ given by (2) belongs to the class S.

Now let $f(z)$ be any function from the class S. We shall show that it can be obtained according to the prescription in (2) with a suitable choice for the function $\mu(x, y)$ from the class $\mathfrak{M}$. To show this let us denote by $\mathfrak{M}'$ the subclass of $\mathfrak{M}$ consisting of those functions $\mu(x, y)$ such that

$$\int_{-\pi}^{\pi} g(w, y)\, d\mu_x'(x, y) = g(w, y(x)).$$

By Löwner's theorem [2] (see also [1], Russian p. 95) the collection of functions $f(z)$ obtained by means of (2) when $\mu(x, y)$ runs over the class $\mathfrak{M}'$, forms a subclass S' of S which is everywhere dense in S with respect to uniform convergence within the circle E.

We select a sequence $f_n(z)$ of functions from the class S' which converges uniformly with respect to E to a function $f(z)$. The sequence $f_n(z)$ corresponds to a sequence $\mu_n(x, y)$ of functions from the class $\mathfrak{M}'$ such that $f_n(z) = \lim_{x\to\infty} e^x f(z, x; \mu_n)$.

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* This follows from the estimate $|f(z, x;\ \mu)| \leqslant |z|$, $|f(z, x;\ \mu)| \leqslant e^{-x}|z| / (1 - |z|)^2$.

From $\mu_n(x, y)$ we can select a subsequence which converges in the sense described above to some function $\mu^*(x, y)$ of the class $\mathfrak{M}$. By using now the assertions of §1, it is not difficult to show that the function $f(z)$ itself can be obtained according to (2) with $\mu = \mu^*$.

It follows from the Riesz-Herglotz theorem [3] that the function

$$h(w, x) = \int_{-\pi}^{\pi} g(w, y)\, d\mu_x'(x, y), \quad \mu \in \mathfrak{M}, \tag{5}$$

is, for each fixed x, $0 < x < \infty$, regular with respect to w in the circle $|w| < 1$ and has a positive real part there. Consequently, from the well-known differential equation of Löwner-Kufarev [4] and from relation (2) we can obtain all the functions of the class S.

3. From the identity

$$dw / dx = -wh(w, x), \quad w = f(z, x; \mu), \tag{6}$$

where $h(w, x)$ is calculated from equation (5), by making use of (2) we immediately obtain the expressions for functions in S

$$f(z) = z \exp\left\{ \int_0^{|z|} \frac{1 - F(w, \rho)}{\operatorname{Re} F(w, \rho)} \frac{d\rho}{\rho} \right\}, \tag{7}$$

$$f'(z) = \exp\left\{ \int_0^{|z|} \frac{1 - F(w, \rho) - wF_w'(w, \rho)}{\operatorname{Re} F(w, \rho)} \frac{d\rho}{\rho} \right\}, \tag{8}$$

which will be necessary for our further considerations.

Here $F(w, \rho) = h(f(z, x(\rho); \mu), x(\rho))$, $\rho = |f(z, x; \mu)|$.*

Theorem 2. *Let z_0 be a fixed point in the circle E and $\alpha, \beta, \gamma, \delta$ be arbitrary real numbers. Then for the functional*

$$I(f) = \alpha \ln\left|\frac{f(z_0)}{z_0}\right| + \beta \arg \frac{f(z_0)}{z_0} + \gamma \ln |f'(z_0)| + \delta \arg f'(z_0) \tag{9}$$

defined over the class S we have the precise estimate

$$\int_0^{|z_0|} \varphi(\xi^-, \eta^-) \frac{d\rho}{\rho} \leqslant I(f) \leqslant \int_0^{|z_0|} \varphi(\xi^+, \eta^+) \frac{d\rho}{\rho} \tag{10}$$

where $(\xi^\pm, \eta^\pm)$ is the point on the curve $\xi^2 - 2a(\rho)\xi + \eta^2 + 1 = 0$, $a = (1 + \rho^2)(1 - \rho^2)^{-1}$ at which the function

$$\varphi(\xi, \eta) = a - \alpha - \gamma + (\alpha + \gamma)/\xi - \gamma\xi - \eta(\delta + (\beta + \delta)/\xi) \tag{11}$$

attains its maximum (minimum) value.

The equality sign in (10) is realized, for example, for functions of the class S having the form $f(z) = \lim_{x \to \infty} e^x f(z, x)$, where $w = f(z, x)$ is a solution to the differential equation $w_x' = -wg(w, y^\pm(x))$, $w(0) = z$, where

$$y^\pm(x) = \arcsin \eta^\pm [\xi^\pm (a^2 - 1)^{1/2}]^{-1} + \int_0^x \eta^\pm dx - \arg z_0,$$

and $\rho = \rho(x)$ is defined by the relation $(\ln \rho)_x' = -\xi^\pm$, $\rho(0) = |z_0|$.

* From (6) it follows that $\rho(x)$ is a monotonically decreasing function, since $(\ln \rho)_x' = -\operatorname{Re} h < 0$.

It follows from formulas (7) and (8) that the problem of estimating a functional $I(f)$ of the form (9) over the class S is equivalent to finding the extrema of the real functional $J(f)=\Psi(h(z), zh'(z))/\mathrm{Re}\,h(z)$, $z=\rho e^{i\phi}\in E$ and is fixed, where $\Psi(\omega, w)=(\alpha+\gamma)(1-\mathrm{Re}\,\omega)-(\beta+\delta)\,\mathrm{Im}\,\omega-\gamma\mathrm{Re}\,w-\delta\,\mathrm{Im}\,w$, on the class P (see in this regard the papers [5,6]), and the subsequent integration of the results over ρ from 0 to $|z_0|$. A similar relation between extremal problems for the classes S and P also holds for other problems in the theory of functions of a complex variable.

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