EXTREMAL PROBLEMS IN CERTAIN CLASSES OF UNIVALENT FUNCTIONS IN A HALF PLANE THAT HAVE FINITE ANGULAR RESIDUES AT INFINITY

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INTRODUCTION

Let $\Pi=\{z\colon \operatorname{Im} z>0\}$ be the upper half plane of the complex plane C_z and $\Pi(\lambda)=\{z\colon \lambda<\arg z<\pi-\lambda\}$, $0<\lambda<\pi/2$, be an angular part of it.

A function f *is said to belong to the class* $\mathscr{R}$ *(is an* $\mathscr{R}$*-function)* if it is holomorphic in the half plane Π and $\operatorname{Im} f(z)\geqslant 0$ for $z\in\Pi$. The following integral representation of functions f of the class $\mathscr{R}$ is well known (see, e.g., [1, pp. 221-222; 2, pp. 519-521; 3, pp. 629-632]):

$$f(z)=\alpha+\beta z+\int\limits_{-\infty}^{\infty}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)d\omega(t), \tag{1}$$

where α and β, $\beta\geqslant 0$, are real constants and $\omega(t)$ is a nondecreasing function in $(-\infty,\infty)$ such that

$$\int\limits_{-\infty}^{\infty}\frac{d\omega(t)}{1+t^2}<\infty.$$

The function $\omega=\omega_f$, occurring in (1), is, in essence, uniquely determined by the $\mathscr{R}$-function f [3]. The numbers α and β in (1) are also uniquely determined with respect to $f\in\mathscr{R}$. For example,

$$(\beta_f=)\,\beta=\lim_{y\to+\infty}\frac{f(iy)}{iy}=\lim_{\substack{z\to\infty\\ z\in\Pi(\lambda)}}\frac{f(z)}{z},$$

and the last limit does not depend on λ, $0<\lambda<\pi/2$.

We fix $\beta\geqslant 0$ and consider the subclass $\mathscr{R}_\beta$ of the functions f of the class $\mathscr{R}$ that can be represented in the form

$$f(z)=\beta z+\int\limits_{-\infty}^{\infty}\frac{d\omega(t)}{t-z},$$

where $\omega(t)$ is a nondecreasing function of bounded variation in $(-\infty,\infty)$. The functions $f\in\mathscr{R}_\beta$ have the following properties: The following equations are valid for $z\to\infty$ over an arbitrary angle $\Pi(\lambda)$, $0<\lambda<\pi/2$:

$$\lim f'(z)=\beta,$$

$$\lim z(\beta z-f(z))=\int\limits_{-\infty}^{\infty}d\omega_f(t).$$

The numbers $\beta=\beta_f$ and $\int\limits_{-\infty}^{\infty}d\omega_f(t)$ are, respectively, called *the angular derivative* and *the angular residue* of f at infinity.

The following proposition, to be used in the sequel, is, in essence, proved in [4].

Proposition 1. If $f\in\mathscr{R}$ and the finite limit

$$\lim z(\beta z-f(z))=\{f\}_1$$

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as $z \to \infty, z \in \Pi(\lambda), 0 < \lambda < \pi/2$, exists for $\beta \geqslant 0$, then $f \in \mathscr{R}_\beta$ and $\int_{-\infty}^{\infty} d\omega_f(t) = \{f\}_1$.

When f runs over the whole class $\mathscr{R}_\beta$, the quantity $\{f\}_1$ runs over the whole interval $[0, \infty)$ and the equality $\{f\}_1 = 0$ is fulfilled if and only if $f(z) = \beta z$.

We fix an arbitrary number $c \geqslant 0$ and denote the class of all the functions $f \in \mathscr{R}_\beta$ whose angular residues $\{f_1\}$ at infinity are equal to c by $\mathscr{R}_\beta(c)$.

Let Q_1 denote the subclass of all the univalent functions of the class $\mathscr{R}_1$ and Q(c) denote the subclass of the univalent functions of the class $\mathscr{R}_1(c)$. Let us observe that the functions of the class Q_1 play an important role in the investigation of planar potential flows, in problems of theory of elasticity, etc.

In [6, 7] a method for the investigation of extremal problems in the classes Q(c), based on the method of inner variations [5], is given. In the present article, we propose another method, based on the parametric representation of functions of the class Q_1. A distinguishing feature of this method, going back in ideological relation to Gutlyanskii's articles [8, 9] and Popov's articles [10, 11], is the reduction of the appropriate extremal problem to a problem of the theory of optimal control. We illustrate the proposed method by complete solution of the problem on the determination of sharp two-sided estimates of the coefficient of inner distortion under the mapping of Π by the functions f of the class Q(c), $c > 0$, depending on the magnitude of the imaginary part of $f(z_0)$ at a preassigned point $z_0 \in \Pi$ (see Theorem 6). The corresponding problem of optimal control is solved by an application of a well-known variant of Pontryagin's maximum principle for a problem "with fixed time." We obtain new results that generalize and refine certain results of [6, 7].

The article [12] contains results that border on the theme of the present article.

1. Extremal Problems in the Class R_0

Let us consider the subclass $R_0 \doteq \mathscr{R}_0(1)$ of the functions p of the class $\mathscr{R}$ that have the integral representation

$$p(z) = \int_{-\infty}^{\infty} \frac{d\nu(t)}{t-z}, \tag{1.1}$$

where $\nu(t)$ is a nondecreasing function in $(-\infty, \infty)$ with the total variation equal to one.

LEMMA 1. For an arbitrarily fixed z_0 such that $\operatorname{Im} z_0 = y_0 > 0$, the set Ω of the values $\zeta = p(z_0)$ on the class R_0 of the functions p is the closed disk $\{\zeta : |\zeta - i/(2y_0)| \leqslant 1/(2y_0)\}$ with the point $\zeta = 0$ deleted.

Proof. The family of the functions $p(z) = (t - z)^{-1}$ that belong to R_0 for real t introduces all the points of the circle $\Gamma = \{\zeta : |\zeta - i/(2y_0)| = 1/(2y_0)\}$, except the point $\zeta = 0$, into Ω. The point $\zeta = 0$ does not belong to Ω, since $\operatorname{Im} p(z_0) > 0$ for $p \in R_0$ and $\operatorname{Im} z_0 > 0$. Since R_0 is convex and the functional $p(z_0)$ is linear, the convex hull of the set $\Gamma \setminus \{0\}$ is contained in Ω. It remains to observe that there is no point of Ω in the exterior of the circle Ω, since, by (1.1),

$$\left| p(z_0) - \frac{i}{2y_0} \right| = \left| \int_{-\infty}^{\infty} \frac{d\nu(t)}{t-z_0} - \frac{i}{2y_0} \int_{-\infty}^{\infty} d\nu(t) \right| \leqslant \frac{1}{2y_0} \int_{-\infty}^{\infty} \left| \frac{t-\bar{z}_0}{t-z_0} \right| d\nu(t) = \frac{1}{2y_0}.$$

The following corollary follows immediately from Lemma 1.

COROLLARY 1. The set of values of $v = \operatorname{Im} p(z_0)$, $\operatorname{Im} z_0 = y_0 > 0$, on the class R_0 of the functions p is the interval $(0, 1/y_0]$. Moreover, for arbitrary fixed $v_0 \in (0, 1/y_0]$, the quantity $u = \operatorname{Re} p(z_0)$, when p runs over the whole set of the functions $p \in R_0$ that satisfy the condition $\operatorname{Im} p(z_0) = v_0$, fills the interval $[-U_0, U_0]$, where $U_0 = (v_0(1/y_0 - v_0))^{1/2}$.

We consider a subclass $R_0^{(n)}$, n = 1, 2,..., of the class R_0 that consists of functions of the form

$$p(z) = \sum_{k=1}^{n} \frac{\varkappa_k}{t_k - z} \tag{1.2}$$

that depend on arbitrary real parameters t_k and $\varkappa_k \geqslant 0$ such that $\varkappa_1 + \varkappa_2 + \ldots + \varkappa_n = 1$. Functions of the form (1.2) are obtained by Eq. (1.1) if we consider $\nu(t)$ as a step function in $(-\infty, \infty)$

with the points of growth $t_1, t_2, \ldots, t_n$.

It is obvious from the proof of Lemma 1 that each point ζ of the set Ω is brought into Ω by a function of the subclass $R_0^{(2)}$, i.e., by a function of the form

$$p(z)=\varkappa/(t_1-z)+(1-\varkappa)/(t_2-z), \tag{1.3}$$

where $0\leqslant\varkappa\leqslant 1$ and t_1 and t_2 are real numbers. In addition, the extreme points of Ω are brought in by functions of the subclass $R_0^{(1)}$.

A remarkable fact is the partial exhibition of a general property intrinsic to the boundary points of the sets of values of the functionals (systems of functionals) that are defined on various classes of analytic functions representable by the Stieltjes integrals (see, e.g., [13]). This property is expressed by the following theorem in relation to the class R_0.

THEOREM 1. Let $\Phi = \Phi(u_0, v_0, u_1, v_1, \ldots, u_n, v_n)$, $n = 0, 1, \ldots$, be a differentiable real-valued function that is defined on the set $G\subset \mathbf{R}^{2(n+1)}$ of values of the system of functionals

$$\{\operatorname{Re} p(z_0),\ \operatorname{Im} p(z_0),\ \operatorname{Re} p'(z_0),\ \operatorname{Im} p'(z_0),\ \ldots,\ \operatorname{Re} p^{(n)}(z_0),\ \operatorname{Im} p^{(n)}(z_0)\}$$

on the class R_0. Here z_0 is a fixed point of Π. Suppose that $\operatorname{grad}\Phi \neq 0$ in G. Then the extremal values of the functional $\Phi(\operatorname{Re} p(z_0), \operatorname{Im} p(z_0), \operatorname{Re} p'(z_0), \operatorname{Im} p'(z_0), \ldots, \operatorname{Re} p^{(n)}(z_0), \operatorname{Im} p^{(n)}(z_0))$ on the whole class R_0 coincide with the extremal values of this functional on the subclass $R_0^{(n+1)}$.

Proof. This theorem is proved in the same manner as [14, Sec. 52, Theorem 3 and Corollary 3].

LEMMA 2. The set of values $\omega = p'(z_0)$, $z_0 = x_0 + iy_0$, $y_0 > 0$, x_0 real, for a fixed value of $\zeta = p(z_0)$ on the subclass $R_0^{(2)}$ of the functions p is the circle $\{\omega\colon |\omega-\zeta^2|=(1-\mu^2)/(4y_0^2)\}$, where $\mu=2y_0|\zeta-i/(2y_0)|\leqslant 1$.

Proof. For functions p of the form (1.3), we have

$$\omega-\zeta^2=\varkappa(1-\varkappa)\left(\frac{1}{t_1-z}-\frac{1}{t_2-z}\right)^2.$$

Representing $\zeta_k = (t_k - z_0)^{-1}$ in the form $\zeta_k = (i + e^{i\alpha_k})/(2y_0)$, where

$$\sin\alpha_k=\frac{y_0^2-(t_k-x_0)^2}{y_0^2+(t_k-x_0)^2},$$

$$\cos\alpha_k=\frac{2y_0(t_k-x_0)}{y_0^2+(t_k-x_0)^2},$$

we get

$$\omega-\zeta^2=\frac{\varkappa(1-\varkappa)}{y_0^2}\exp\{i(\alpha_1+\alpha_2+\pi)\}\sin^2\frac{\alpha_1-\alpha_2}{2}.$$

Let $\mu=2y_0|\zeta-i/(2y_0)|$. By Lemma 1, we have $\mu\leqslant 1$. Direct computation of μ by using the equation $\zeta=i/(2y_0)+\varkappa\exp(i\alpha_1)+(1-\varkappa)\exp(i\alpha_2)$ leads to

$$\varkappa(1-\varkappa)\sin^2\frac{\alpha_1-\alpha_2}{2}=\frac{1-\mu^2}{4}.$$

Therefore,

$$\omega-\zeta^2=\frac{1-\mu^2}{4y_0^2}e^{i(\alpha_1+\alpha_2+\pi)}.$$

Moreover, since t_1 and t_2 are arbitrary real numbers, the quantity $\alpha_1 + \alpha_2 + \pi$ can take arbitrary values from the interval $[0, 2\pi]$. The lemma is proved.

The following theorem follows from Theorem 1 and Lemmas 1 and 2.

THEOREM 2. Let the functional

$$\mathcal{I}(p)=\Phi(\operatorname{Re} p(z_0), \operatorname{Im} p(z_0), \operatorname{Re} p'(z_0), \operatorname{Im} p'(z_0)),$$

where $\operatorname{Im} z_0 = y_0 > 0$ and $\Phi(u_0, v_0, u_1, v_1)$ is a differentiable function such that $\operatorname{grad}\Phi \neq 0$, be defined on the class R_0 of functions p. Then the extremum of $\mathcal{I}(p)$ on the whole class R_0 coincides with the extremum of $\mathcal{I}(p)$ on the subclass $R_0^{(2)}$, and, in addition,

$$\operatorname{extr}_{p\in R_0} \mathcal{J}(p) = \operatorname{extr}_{\zeta} \operatorname{extr}_{\omega} \Phi(\operatorname{Re}\zeta, \operatorname{Im}\zeta, \operatorname{Re}\omega, \operatorname{Im}\omega),$$

where ζ varies in the set $\{\zeta : |\zeta - i/(2y_0)| \leqslant 1/(2y_0)\}\backslash\{0\}$ and ω varies (for fixed ζ) on the circle

$$\{\omega: |\omega - \zeta^2| = (1-\mu^2)/(4y_0^2)\}, \quad \mu = 2y_0 |\zeta - i/(2y_0)|.$$

COROLLARY 2. The following sharp (in the class R_0) inequalities hold for an arbitrarily fixed value of $v_0 = \operatorname{Im} p(z_0)$, $v_0 \in (0, 1/y_0]$, $\operatorname{Im} z_0 = y_0 > 0$, $p \in R_0$:

$$-v_0/y_0 \leqslant \operatorname{Re} p'(z_0) \leqslant v_0(1/y_0 - 2v_0), \tag{1.4}$$

$$|\operatorname{Im} p'(z_0)| \leqslant \begin{cases} v_0/y_0 & \text{for} \quad 0 < v_0 \leqslant 1/2y_0, \\ 2v_0^{3/2}(1/y_0 - v_0)^{1/2} & \text{for} \quad 1/2y_0 \leqslant v_0 \leqslant 1/y_0. \end{cases} \tag{1.5}$$

Indeed, by Theorem 2, it is sufficient to prove estimates (1.4) and (1.5) for functions of the subclass $R_0^{(2)}$. Expressing $\zeta = p(z_0) = u_0 + iv_0$, $\omega = p'(z_0)$, $p \in R_0^{(2)}$, in the form

$$\zeta = \frac{i}{2y_0} + \frac{\mu}{2y_0} e^{i\alpha},$$

$$\omega = \zeta^2 + \frac{1-\mu^2}{4y_0^2} e^{i\beta},$$

where $\mu \in [0, 1]$, we get $u_0 = \mu \cos\alpha/(2y_0)$ and $v_0 = 1/(2y_0) + \mu \sin\alpha/(2y_0)$. Therefore,

$$\operatorname{Re}\omega = \frac{v_0}{y_0} - 2v_0^2 - \frac{1-\mu^2}{4y_0^2}(1-\cos\beta), \tag{1.6}$$

$$\operatorname{Im}\omega = 2u_0 v_0 + \frac{1-\mu^2}{4y_0^2}\sin\beta. \tag{1.7}$$

The inequality $\mu \geqslant 2y_0|v_0 - 1/(2y_0)|$ is fulfilled for fixed v_0, $v_0 \in (0, 1/y_0]$. By (1.6), the greatest value of $\operatorname{Re}\omega$, equal to $v_0/y_0 - 2v_0^2$, is attained for $\beta = 0$ and the least value, equal to $-v_0/y_0$, is attained for $\mu = 2y_0|v_0 - 1/(2y_0)|$ and $\beta = \pi$. By (1.7), we have

$$|\operatorname{Im}\omega| \leqslant 2|u_0|v_0 + (1-\mu^2)/4y_0^2 = v_0/y_0 - (v_0 - |u_0|)^2.$$

Now, since $|u_0| \leqslant U_0$ by Corollary 1 and $|u_0|$ can be an arbitrary element of the interval $[0, U_0]$, where $U_0 = (v_0(1/y_0 - v_0))^{1/2}$, we find the minimum of $v_0 - |u_0|$ in the indicated interval. This minimum turns out to be equal to zero if $0 < v_0 \leqslant 1/(2y_0)$ and is equal to $v_0 - U_0$ if $1/(2y_0) \leqslant v_0 \leqslant 1/y$.

2. Parametric Representation of Sets of Values of Certain Functionals on the Class Q(c)

Let $\gamma > 0$ and $\mathfrak{N}_\gamma$ be the family of all the real-valued functions $\nu_\tau = \nu_\tau(t)$ that are measurable with respect to the parameter τ, $\tau \in [0, \gamma]$, are nondecreasing with respect to t in $(-\infty, \infty)$ for almost all fixed $\tau \in [0, \gamma]$ with total variation in $(-\infty, \infty)$ with respect to t equal to one and the normalizing condition $\nu_\tau(-\infty) = \lim_{t\to-\infty} \nu_\tau(t) = 0$.

Let $R(R_0, T)$, $T = [0, c]$, $c > 0$, be the set of all the functions $h(\zeta, \tau)$ that are defined on $\Pi \times T$ by the formula

$$h(\zeta, \tau) = \int_{-\infty}^{\infty} \frac{d\nu_\tau(t)}{t - \zeta}, \quad \nu_\tau \in \mathfrak{N}_c. \tag{2.1}$$

For almost each fixed $\tau \in T$, each function $h(\zeta, \tau) \in R(R_0, T)$ is a function of the class R_0 with respect to ζ.

Let us consider the differential equation

$$\frac{d\zeta}{d\tau} = h(\zeta, \tau) \tag{2.2}$$

with the right-hand side $h(\zeta, \tau) \in R(R_0, T)$. Let $\zeta = \zeta(z, \tau)$ denote the solution of this equation in the sense of Caratheodory [i.e., a function for which (2.2) is fulfilled a.e. in T] that satisfies the initial condition $\zeta|_{\tau=0} = z$, $z \in \Pi$.

It has been shown in [5] that for each $h(\zeta, \tau) \in R(R_0, T)$ the solution $\zeta(z, \tau)$ of the indicated Cauchy problem for Eq. (2.2) exists and is unique, and $\zeta(\cdot, \tau) \in Q(\tau)$ for almost all $\tau \in T$. The following theorem is also valid.

THEOREM 3 [15]. Each function $f(z) \not\equiv z$ of class Q_1 can be represented in the form $f(z) = \zeta(z, c)$, where $\zeta(z, \tau)$ is the solution of the Cauchy problem

$$\frac{d\zeta}{d\tau} = \int_{-\infty}^{\infty} \frac{d\nu_\tau(t)}{t-\zeta}, \tag{2.3}$$

$$\zeta|_{\tau=0} = z, \quad z \in \Pi, \tag{2.4}$$

with a certain function $\nu_\tau(t) \in \mathfrak{N}_c$ and $c = \{f_1\}$ is the angular residue of f at infinity.

Now let f be an arbitrary function of class Q_1 that is not the identity function ($f(z) \not\equiv z$), and $c = \{f_1\}$, $\nu_\tau(t)$ be a function of the family $\mathfrak{N}_c$ that generates f by the method described in Theorem 3, and $\zeta(z, \tau)$ be the solution of the Cauchy problem (2.3), (2.4). We fix an arbitrary point $z_0 \in \Pi$ and set $\zeta(z_0, \tau) = \xi(\tau) = \xi$, $\operatorname{Re}\xi(\tau) = x(\tau) = x$, and $\operatorname{Im}\xi(\tau) = y(\tau) = y$. The function $\xi(\tau) = x(\tau) + iy(\tau)$ is defined on $[0, c]$ and is such that $\xi(0) = z_0$ and $\xi(c) = f(z_0)$, and for almost all $\tau \in [0, c]$

$$\frac{d\xi}{d\tau} = \int_{-\infty}^{\infty} \frac{d\nu_\tau(t)}{t-\xi}. \tag{2.5}$$

From (2.5) we have

$$\frac{1}{y}\cdot\frac{dy}{d\tau} = \int_{-\infty}^{\infty} \frac{d\nu_\tau(t)}{|t-\xi|^2}, \tag{2.6}$$

whence it is obvious that $y = y(\tau)$ is an increasing function on $[0, c]$.

Let us consider the function

$$p(\zeta, \tau) = y(\tau) \int_{-\infty}^{\infty} \frac{d\nu_\tau(t)}{t - y(\tau)\zeta - x(\tau)}.$$

For almost all $\tau \in [0, c]$ it is an $\mathscr{R}$-function of ζ with the angular derivative at infinity equal to zero and the angular residue at infinity equal to

$$-\lim_{\substack{\zeta\to\infty \\ \zeta\in\Pi(\lambda)}} \zeta p(\zeta, \tau) = \int_{-\infty}^{\infty} d\nu_\tau(t) = 1 \quad (0 < \lambda < \pi/2).$$

By supposition (see Introduction), $p(\cdot, \tau) \in R_0$ for almost all $\tau \in [0, c]$.

We write Eq. (2.3) in the form

$$\frac{d\zeta}{d\tau} = \frac{1}{y} p\left(\frac{\zeta - x}{y}, \tau\right). \tag{2.7}$$

Setting here $z = z_0$, we get

$$\frac{d\xi}{d\tau} = \frac{1}{y} p(i, \tau). \tag{2.8}$$

Let us set $\operatorname{Im} p(i, \tau) = v(\tau) = v$. By Corollary 1, $0 < v \leqslant 1$ a.e. in $[0, c]$. From (2.8) we have

$$y\,dy = v(\tau)\,d\tau, \tag{2.9}$$

whence, by virtue of the conditions $y(0) = \operatorname{Im} z_0 = y_0$ and $y(c) = \operatorname{Im} f(z_0)$, on integration we get

$$(\operatorname{Im} f(z_0))^2 = y_0^2 + 2\int_0^c v(\tau)\,d\tau \leqslant y_0^2 + 2c. \tag{2.10}$$

Considering the family of the functions

$$\varphi_a(z) = a + \sqrt{(z-a)^2 - 2c}, \tag{2.11}$$

that belong (for a suitable choice of branch of the radical) to the class Q(c), $c > 0$, for each real a we verify that for each number A in the interval $(1, (1 + 2cy_0^{-2})^{1/2}]$ there exists a such that $\mathrm{Im}\,\varphi_a(z_0)/\mathrm{Im}\,z_0 = A$. Therefore, the above interval is entirely contained in the set of values of $\mathrm{Im}\,f(z_0)/\mathrm{Im}\,z_0$ on the class Q(c), $c > 0$, of the functions f. Together with the estimate (2.10), this leads to the following theorem.*

THEOREM 4. The interval $(1, I_0]$, where $I_1^0 = I_1^0(c, y_0) = (1 + 2cy_0^{-2})^{1/2}$, is the set of values of $\mathrm{Im}\,f(z_0)/\mathrm{Im}\,z_0$ for fixed $\mathrm{Im}\,z_0 = y_0 > 0$ on the class Q(c), $c > 0$, of the functions f.

Now let z_0 be a fixed point of Π, $\mathrm{Im}\,z_0 = y_0$, and y_1 be an arbitrary fixed point in $(y_0, y_0 I_1^0(c, y_0)]$. Let us consider the class $Q(c, y_1)$ of all the functions $f \in Q(c)$ such that $\mathrm{Im}\,f(z_0) = y_1$.

THEOREM 5. Let the functional

$$\mathcal{I}(f) = \Phi(\mathrm{Re}\,f(z_0),\ \mathrm{Re}\,f'(z_0),\ \mathrm{Im}\,f'(z_0)),$$

where $\Phi(u_0, u_1, v_1)$ is a real-valued function, be defined on the class $Q(c, y_1)$ of the functions f. Then the set of values of $\mathcal{I}(f)$ on $Q(c, y_1)$ coincides with the set of values of $\Phi(\mathrm{Re}\,w_0, \mathrm{Re}\,w_1, \mathrm{Im}\,w_1)$, where

$$w_0 = z_0 + \int_0^c p(i, \tau) \frac{d\tau}{y(\tau)}, \tag{2.12}$$

$$w_1 = \exp \int_0^c p'_\zeta(i, \tau) \frac{d\tau}{y^2(\tau)}, \tag{2.13}$$

$y(\tau)$ being the solution of the equation $y\,dy = v(\tau)d\tau$ for almost all $\tau \in [0, c]$, $v(\tau) = \mathrm{Im}\,p(i, \tau)$, with the condition $y(0) = y_0 = \mathrm{Im}\,z_0$, when $p(\zeta, \tau)$ runs over the family of all the functions $y(\tau)$ from $R(R_0, T)$, $T = [0, c]$, such that $y(c) = y_1$.

By virtue of what we have said above, to prove Theorem 5 it is sufficient to note that the equation

$$d \ln \zeta'_z(z_0, \tau) = \frac{1}{y^2(\tau)} p'_\zeta(i, \tau)\, d\tau,$$

which follows from (2.7), is fulfilled a.e. in [0, c] for the solution $\zeta(z, \tau)$ of the Cauchy problem (2.3), (2.4) that generates an arbitrary preassigned function $f \in Q(c, y_1)$ by the method described in Theorem 3. Integrating the above equation and (2.8) and taking into account the conditions $\xi(0) = z_0$, $\xi(c) = f(z_0)$, $\zeta'_z(z_0, 0) = 1$, and $\zeta'_z(z_0, c) = f'(z_0)$, we get the expressions (2.12) and (2.13) for the quantities $w_0 = f(z_0)$ and $w_1 = f'(z_0)$.

3. A Theorem on the Relative Inner Distortion under Mappings by Functions of the Class Q(c)

THEOREM 6. The set of values of the system $I(f) = \{I_1, I_2\}$ of the functionals

$$I_1 = I_1(f) = \frac{\mathrm{Im}\,f(z_0)}{\mathrm{Im}\,z_0}, \quad I_2 = I_2(f) = \ln|f'(z_0)|, \tag{3.1}$$

$$\mathrm{Im}\,z_0 = y_0 > 0,$$

on the class Q(c), $c > 0$, of the functions f is determined by the inequalities

$$1 < I_1 \leqslant I_1^0,$$

$$-\ln I_1 \leqslant I_2 \leqslant \begin{cases} \ln I_1 - 2c^{-1}y_0^2(I_1 - 1)^2 & \text{for } 1 < I_1 \leqslant I_1', \\ \ln \dfrac{\left(1 + \left(I_1^{0\,2} - I_1^2\right)^{1/2}\right)^2}{I_1} - 2\dfrac{\left(I_1^{0\,2} - I_1^2\right)^{1/2}}{1 + \left(I_1^{0\,2} - I_1^2\right)^{1/2}} & \text{for } I_1' \leqslant I_1 \leqslant I_1^0, \end{cases}$$

where $I_1^0 = (1 + 2cy_0^{-2})^{1/2}$ and $I_1' = (1 + (1 + 4cy_0^{-2})^{1/2})/2$.

Proof. The inequalities for I_1 have been proved above. We take an arbitrary $I_1 \in (1, I_1^0]$ and estimate $I_2(f)$ on the class $Q(c, y_1)$, $y_1 = y_0 I_1$, of the functions f. By Theorem 5 and Corollaries 1 and 2, we have

*The result of Theorem 4 was obtained earlier in [16] by another method.

$$-\inf \int_0^c v \frac{d\tau}{y^2} \leqslant I_2(f) \leqslant \sup \int_0^c v(1-2v)\frac{d\tau}{y^2}, \tag{3.2}$$

where $y = y(\tau)$ satisfies Eq. (2.9) a.e. in $[0, c]$ and the condition $y(0) = y_0$ and the infimum and the supremum are taken over all possible functions $v = v(\tau)$ that take values in the interval $(0, 1]$ for almost all $\tau \in [0, c]$ and are such that $y(c) = y_1$. Making the change of variable $\tau = \tau(y)$ in the left integral in (3.2), by virtue of (2.9) we get

$$\int_0^c v\frac{d\tau}{y^2} = \int_{y_0}^{y_1} \frac{dy}{y} = \ln I_1,$$

which proves the lower estimate of I_2 in the statement of the theorem.

We formulate the problem on the upper estimate of I_2, reducing to the determination of supremum in (3.2), in terms of the theory of optimal control in the following manner.

Find, among all admissible controls $v = v(\tau)$, $0 \leqslant v \leqslant 1$, under the action of which on the interval $[0, c]$ ($c > 0$ is given) the phase coordinate $y = y(\tau)$ transforms from a given initial state $y(0) = y_0$ into a prescribed final state $y(c) = y_1$ along the trajectory of the differential equation (2.9), a control for which the integral

$$\int_0^c v(1-2v)\frac{d\tau}{y^2}$$

takes the greatest possible value.

To solve the formulated problem, we use a well-known variant of Pontryagin's maximum principle for the problem "with fixed time" [17, pp. 373-374]. In the present case, the Pontryagin function $H(\Psi, y, v)$ has the form

$$H(\Psi, y, v) = \frac{v-2v^2}{y^2} + \Psi\frac{v}{y},$$

where $\Psi = \Psi(\tau)$ is the solution of the equation

$$\frac{d\Psi}{d\tau} = \frac{2(v-2v^2)}{y^3} + \Psi\frac{v}{y^2}.$$

The condition for maximum is fulfilled if $v = v^* = \min\{1, (1+\Psi y)/4\}$, $\Psi y > -1$.

For $v = v^* = 1$

$$\begin{cases} \dfrac{dy}{d\tau} = \dfrac{1}{y}, \\ \dfrac{d\Psi}{d\tau} = \dfrac{-2+\Psi y}{y^3}, \end{cases}$$

whence $y^2 = 2\tau + \text{const}$ and $\Psi y = k_1 y^2 + 1$, $k_1 = \text{const}$. Since $\Psi y \geqslant 3$ for $v^* = 1$, it follows that $k_1 > 0$ and Ψy is a continuous increasing function of τ.

We show that the control $v = 1$ is optimal on the whole interval $[0, c]$ if and only if I_1 takes the greatest possible value $I_1 = I_1^0$. Indeed, the equation $I_1 = I_1^0$ is equivalent to the equation $2c = y_1^2 - y_0^2$, which, by virtue of the condition

$$2\int_0^c v\,d\tau = y_1^2 - y_0^2,$$

which follows from (2.9) and the conditions $y(0) = y_0$ and $y(c) = y_1$, is fulfilled under the restrictions $0 < v \leqslant 1$ if and only if $v = 1$ a.e. in $[0, c]$. In addition,

$$-\inf\int_0^c v\frac{d\tau}{y^2} = \sup\int_0^c v(1-2v)\frac{d\tau}{y^2} = -\int_0^c \frac{d\tau}{y^2} = -\int_{y_0}^{y_1}\frac{dy}{y} = -\ln I_1^0.$$

Thus, for $I_1 = I_1^0$ the functional $I_2(f)$ takes the single value $-\ln I_1^0$ on the class $Q(c)$ of the functions f.

Let us consider the case $I_1 \in (1, I_1^0)$. Here the control v = 1 cannot be optimal on the whole interval [0, c]. But then, as shown above, the control $v = (1 + \Psi y)/4$ must be optimal on a certain part of the interval [0, c]. For this control we have

$$\begin{cases} \dfrac{dy}{d\tau} = \dfrac{1+\Psi y}{4y} > 0, \\ \dfrac{d\Psi}{d\tau} = \dfrac{1+\Psi y}{4y^3}, \end{cases}$$

whence $\Psi y = 4k_2 y - 1$, where k_2 is a positive constant and $y = k_2\tau + \text{const}$. In addition, $\Psi y \leqslant 3$ and Ψy and $v = k_2 y$ are increasing functions of τ.

Let us suppose that the control $v = (1 + \Psi y)/4$ is optimal on the whole interval [0, c]. Then $y = k_2\tau + y_0$ and $k_2 = (y_1 - y_0)/c$. The condition $v(c) \leqslant 1$ reduces to the inequality $y_1 \times (y_1 - y_0) \leqslant c$, which is fulfilled only when $I_1 \in (1, I_1']$ (I_1' is indicated in the statement of the theorem). It is easy to verify the converse: For $I_1 \in (1, I_1']$ the control $v = k_2 y$ is optimal on the whole interval [0, c]. In addition,

$$\sup I_2(f) = \int_{y_0}^{y_1} \left(1 - 2\frac{y_1 - y_0}{c}\right)\frac{dy}{y} = \ln I_1 - 2c^{-1}y_0^2(I_1 - 1)^2.$$

Let us pass to the consideration of the case $I_1 \in (I_1', I_1^0)$. Here the control v = 1 is optimal on a part of the interval [0, c] and the control $v = k_2 y = k_2(k_2\tau + \text{const}) < 1$, where k_2 is a positive constant, is optimal on the remaining part of [0, c]. Let τ^*, $\tau^* \in (0, c)$ be a switching point of the control. We have $\Psi y = 3$ at this point, whereas when τ passes through the point τ^* from left to right the sign of $\Psi y - 3$ changes in the opposite direction. From the above-established form of the function $\Psi(y)$ in the left and right halfneighborhoods of τ^* we conclude that v cannot be equal to 1 on the left of τ^*. Consequently, the switching point of the control in [0, c] is unique and the regular synthesis of the optimal control can be written in the form

$$v = v(\tau) = \begin{cases} k_2 y & \text{for} \quad 0 \leqslant \tau < \tau^*, \\ 1 & \text{for} \quad \tau^* \leqslant \tau \leqslant c, \end{cases}$$

where k_2 = const. Determining k_2 from the condition $v(\tau^* - 0) = 1$, we get $k_2 = 1/y(\tau^* - 0)$. Consequently, $y(\tau^* - 0) = y_0 + \tau^*/y(\tau^* - 0)$. On the other hand, $y^2(\tau^* + 0) = y_1^2 + 2(\tau^* - c)$. Since $y(\tau)$ is continuous for $\tau = \tau^*$, as also everywhere in [0, c] [this follows from (2.9) and the form of $v(\tau)$ found by us], we have

$$y^* = y(\tau^*) = (1 + \Delta)y_0, \quad \Delta = \left(I_1^{0^2} - I_1^2\right)^{1/2}.$$

In addition

$$\sup I_2(f) = \frac{1}{y^*}\int_0^{\tau^*} \left(1 - 2\frac{y}{y^*}\right)\frac{d\tau}{y} - \int_{\tau^*}^{c} \frac{d\tau}{y^2} = \int_{y_0}^{y^*} \left(1 - 2\frac{y}{y^*}\right)\frac{dy}{y} - \int_{y^*}^{y_1} \frac{dy}{y} = \ln\frac{(1+\Delta)^2}{I_1} - 2\frac{\Delta}{1+\Delta}.$$

The theorem is proved.

Let $D = D(c, y_0)$ be the set of values of the system $I(f) = \{I_1, I_2\}$ of the functionals I_1 and I_2 of the form (3.1), $\operatorname{Im} z_0 = y_0 > 0$, on the class Q(c), c > 0, of the functions f. It is obvious from Theorem 6 that D depends only on the ratio c/y_0^2. Let $\lambda = cy_0^{-2}$ and $D(\lambda) = D(c, y_0)$. Figure 1 gives a graphical representation of the form of $D(\lambda)$ and the dependence of $D(\lambda)$ on λ.

An elementary investigation of the equations of the boundary $\Gamma(\lambda)$ of the set $D(\lambda)$ shows that $D(\lambda)$ is a convex set with boundary that is smooth everywhere, except the points (1; 0) and $(I_1^0; -\ln I_1^0)$, at which the boundary arcs meet at angles $\pi/2$ and $\arctan[(1 + 2\lambda)^{3/2}(2 + 3\lambda)^{-1}]$, respectively. We have $D(\lambda_1) \subset D(\lambda_2)$ for $0 < \lambda_1 < \lambda_2$.

The set $D(\lambda)$, $\lambda > 0$, is not closed: Its boundary point (1; 0) does not belong to it. All the remaining points of the boundary $\Gamma(\lambda)$ belong to $D(\lambda)$. The points of the lower part of the boundary $\Gamma(\lambda)$ with the equation $I_2 = -\ln I_1$, $1 < I_1 \leqslant I_1^0$, $I_1^0 = I_1^0(\lambda) = (1 + 2\lambda)^{1/2}$, are attained in $D(\lambda)$ by functions of the form (2.11).

Remark. The following inequalities are valid for each point $(I_1; I_2) \in D(\lambda)$, $\lambda > 0$:

$$1 < I_1 \leqslant I_1^0, \tag{3.3}$$

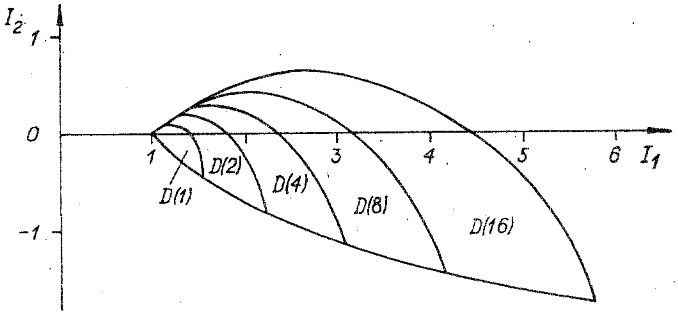

Fig. 1

$$-\ln I_1^0 \leqslant I_2 \leqslant I_2^0, \tag{3.4}$$

where $I_2^0 = I_2^0(\lambda) = \sigma - \ln(2\sigma) - 0{,}5$, $\sigma = \sigma(\lambda) = ((1+\lambda)^{1/2} - 1)/\lambda$, and I_1^0 is indicated above. Estimates (3.3), (3.4) are equivalent to the indication of the rectangle in the plane with the rectangular system of coordinates I_1, I_2 and with sides that are parallel to coordinate axes which majorizes $D(\lambda)$ and touches the boundary $\Gamma(\lambda)$ at the points (1; 0), (I_1^0; $-\ln I_1^0$), and ($(1 + (1 + \lambda)^{1/2})/2$; I_2^0). The estimates (3.4) were obtained earlier in [6] by another method.

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STARLIKENESS OF FUNCTIONS WITH BOUNDED MEAN MODULUS

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1. INTRODUCTION

Let H_δ ($\delta > 0$) be the class of the functions that are regular in the disk $|z| < 1$ and satisfy the condition

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta \leqslant 1, \quad r = |z|, \tag{1}$$

and let $H_\delta^m(c_m)$, $m = 0, 1, \ldots$, denote the subclass of functions from H_δ that do not vanish for $z \neq 0$ and the coefficient c_m in whose expansion

$$f(z) = c_m z^m + c_{m+1} z^{m+1} + \ldots$$

is fixed.

The class H_δ is a subclass, introduced by Hardy, of the class of the functions with bounded mean modulus and is the set of the functions $f(z)$ that are regular in the unit disk and are characterized by the condition

$$\int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta < +\infty.$$

The following integral representation of Smirnov [1] for the above functions is well known:

$$f(z) = B(z) \exp \frac{1}{2\pi}\int_0^{2\pi} g(z, \varphi) \ln p(\varphi) \, d\varphi \exp \frac{1}{2\pi}\int_0^{2\pi} g(z, \varphi) \, dq(\varphi), \tag{2}$$

where $B(z)$ is the so-called Blaschke function that has all the zeros of $f(z)$, $g(z, \varphi) = (e^{i\varphi} + z)/(e^{i\varphi} - z)$, $p(\varphi)$ is a nonnegative function such that $\ln p(\varphi)$ and $p(\varphi)^\delta$ are summable, and $q(\varphi)$ is a nonincreasing function with the derivative equal to zero almost everywhere.

The conditions $B(z) = e^{i\mu} z^m$, $\mu = \arg c_m$,

$$\frac{1}{2\pi}\int_0^{2\pi} p(\varphi)^\delta \, d\varphi \leqslant 1, \tag{3}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \ln p(\varphi) \, d\varphi + \frac{1}{2\pi}\int_0^{2\pi} dq(\varphi) = \ln |c_m| \tag{4}$$

select the subclass $H_\delta^m(c_m)$ from the functions that are represented by Eq. (2).

The starlikeness of a class of bounded functions, an immediate generalization of which is H_δ, has been investigated by Goluzin [2]. The radius of starlikeness in a class of functions of bounded form that contains H_δ in it has been obtained in [3]. In the class H_δ itself, the exact value of the radius of starlikeness is yet unknown, although attempts to find its bounds or to find its exact value for certain subclasses of H_δ were made in [4-6]. The aim of the present article is to fill this gap.

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