Applications of the Schwarzian Derivative

A new type of derivative? The Schwarzian appears in complex analysis, differential equations, and one-dimensional dynamics. More recently, we also see it in Teichmüller theory, integrable systems, and conformal field theory. What it is, what it does to you and how to defend yourself if it gets you!

Helge Ruddat

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What is a Schwarzian ?

Definition: The Schwarzian derivative of $f(z)$ is:

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

it equals

$$\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

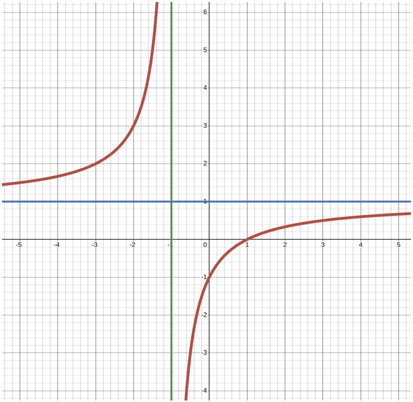
also equals

$$\frac{\varphi''}{\varphi} \quad \text{for} \qquad \varphi = \frac{1}{\sqrt{f'}}.$$

The last was its first appearance in Lagrange's essay *"Sur la construction des cartes geographiques"*, 1781.

Example: A Typical Möbius Transformation

$f(x) = \frac{x-1}{x+1}$



A Möbius transformation is a map of the form $f(x) = \frac{ax+b}{cx+d}$ (so here we used $a = 1, b = -1, c = 1, d = 1$).

Fact: f is a Möbius transformations $\iff S(f) = 0$.

Applications in complex analysis

Studies properties of conformal mappings in the complex plane (cartography).

Analyzes boundary behavior in the Riemann mapping theorem.

Nehari theorem: If $f(z)$ is analytic in $\mathbb{D} = \{z : |z| < 1\}$ and:

$$|S(f)(z)| \leq \frac{2}{(1 - |z|^2)^2}, \quad \text{for all } z \in \mathbb{D},$$

then f is univalent (=injective) in $\mathbb{D}$.

Quantifies deviation from being a Möbius transformation; classifies mappings in geometric function theory.

Differential Equations (Sturm-Liouville Theory)

Given some potential $q(x)$, e.g., $q(x) = \textit{const}$ for the harmonic oscillator, the equation

$$u''(x) + q(x)u(x) = 0$$

for a function $u(x)$ has a two-dimensional space of solutions.

If $u_1(x)$ and $u_2(x)$ are any two linearly independent solutions and $f = u_1/u_2$ is their ratio, we can reproduce the potential from them:

$$q = \frac{1}{2}S(f).$$

Further applications

Analyzes stability and chaos in iterated functions $f^n(z)$, here $S(f) < 0$ implies simpler dynamics (at most one attracting fixed point).

Relates to projective structures and uniformization in Riemann surfaces: Connects complex analysis to hyperbolic geometry.

Virasoro algebra in conformal field theory (CFT).

AdS/CFT correspondence and black hole physics (e.g., SYK model).

Influences trajectories and stability in optimization problems.

Designs systems with geometric constraints.

Assesses quality of approximations for functions with rapid curvature changes.

Refines algorithms for smooth reconstruction.

Is $S(f)$ a derivative ?

How does $S(f)$ transform under coordinate changes ?

If $f(x)$ is a function and y is a new coordinate, we get $f(y)$ simply be inserting $x(y)$ into $f(x)$:

$$f(y) = f(x(y)).$$

On the other hand, the derivative f' of f transforms differently, we need to use the chain rule:

$$f'(y) = f'(x(y))x'(y)$$

The invariant geometric quantity is not the derivative but the differential: $df = f'(x)dx$ The geometric invariant quantity corresponding to $\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ is the **quadratic differential**:

$$S(f) = S(f(x))(dx)^2$$

Measuring infinitesimal changes of cross-ratios

Given four points $t_1, t_2, t_3, t_4 \in \mathbb{R}$, their cross ratio

$$[t_1, t_2, t_3, t_4] := \frac{(t_1 - t_3)(t_2 - t_4)}{(t_1 - t_2)(t_3 - t_4)}$$

is invariant under Möbius transformations (does not change under them). Now consider any diffeomorphism $f : \mathbb{R} \to \mathbb{R}$, let $t \in \mathbb{R}$ be a point and v a tangent vector at t, let ϕ_s be the local flow and consider the four points

$$t, t_1 = \phi_\epsilon(t), t_2 = \phi_{2\epsilon}(t), t_3 = \phi_{3\epsilon}(t)$$

then

$$[f(t), f(t_1), f(t_2), f(t_3)] = [t, t_1, t_2, t_3] - 2\epsilon^2 S(f)(t) + O(\epsilon^3).$$

Thurston's Interpretation of the Schwarzian Derivative

William Thurston interprets the Schwarzian derivative as a measure of how much a conformal map f deviates from a Möbius transformation.

Let f be a conformal mapping in a neighborhood of $z_0 \in \mathbb{C}$. Then there exists a unique Möbius transformation M such that M, f have the same 0, 1, 2-th order derivatives at z_0. Now:

$$(M^{-1} \circ f)(z - z_0) = z_0 + (z - z_0) + \frac{1}{6} a(z - z_0)^3 + \cdots .$$

To solve for a, consider $z_0 = 0$. Let:

$$M^{-1}(z) = \frac{Az + B}{Cz + 1},$$

and solve for A, B, C such that the first three coefficients of $M^{-1} \circ f$ equal $z_0, 1, 0$.

Plugging into the fourth coefficient, we get:

$$a = (S(f))(z_0).$$