STABILITY OF A COMPRESSIBLE TWO-FLUID HYPERBOLIC-ELLIPTIC SYSTEM ARISING IN FLUID MECHANICS

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Abstract. This paper deals with an initial-boundary value problem for the following one-dimensional two-fluid system

$$\begin{cases} n_t + (nu_g)_x = 0, & x \in I = [0,1], \quad t > 0, \\ m_t + (mu_l)_x = 0, & \\ \alpha_g [P_g]_x = \mu_g [u_g]_{xx}, & \\ \alpha_l [P_l]_x = \mu_l [u_l]_{xx}, & \alpha_l + \alpha_g = 1, \end{cases}$$

where n and m represent, respectively, gas mass and liquid mass; u_g and u_l are corresponding fluid velocities whereas α_g and α_l are volume fractions occupied by the gas and liquid phase, and P_g and P_l are pressures associated with them. The model represents a submodel of the full two-fluid model studied in [5]. An important difference between the model studied in the present work and that studied in [5] is that viscosity coefficients μ_l, μ_g are assumed to be constant. Bresch et al assumed mass-dependent coefficients that allowed them to derive a so-called BD inequality which implies that masses are in H^1. Since we are excluded from following that route, we instead explore how the use of two non-equal pressure functions P_g and P_l (i.e., $P_l - P_g = f(m) \neq 0$) allows us to obtain global estimates that guarantee a stability result to hold. I.e., we prove that

$$m(\cdot,t) \to \tilde{m}, \qquad n(\cdot,t) \to \tilde{n}, \qquad u_l(\cdot,t), u_g(\cdot,t) \to 0,$$

with respect to the norm in $L^\infty(I)$ for constant states $\tilde{m}$ and $\tilde{n}$. Estimates of the time asymptotic behavior are also provided.

Keyword: two-fluid model, non-equal pressure, capillary pressure, Navier-Stokes, existence, stability
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1. Introduction

This paper deals with a mathematical model for gas-liquid flow dynamics where the gas phase is modelled as an ideal gas whereas the liquid phase is assumed to be weakly compressible. The model is based on the so-called two-fluid formulation where the gas and liquid phase have separate mass and momentum conservation equations. In particular, the momentum equations involve a non-conservative pressure-related term, a viscous term and external force terms representing gravity and friction between fluid and wall as well as interfacial friction. The model takes the following form [18] (Chapter 10):

$$\begin{aligned} \partial_t(n) + \partial_x(nu_g) &= 0 \\ \partial_t(m) + \partial_x(mu_l) &= 0 \\ \partial_t(nu_g) + \partial_x(nu_g^2) + \alpha_g \partial_x P_g &= -f_g u_g - I(u_g - u_l) - ng + \partial_x(\mu_g \partial_x u_g) \\ \partial_t(mu_l) + \partial_x(mu_l^2) + \alpha_l \partial_x P_l &= -f_l u_l + I(u_g - u_l) - mg + \partial_x(\mu_l \partial_x u_l). \end{aligned} \tag{1.1}$$

Here $n = \alpha_g \rho_g$ and $m = \alpha_l \rho_l$ where the volume fractions satisfy

$$\alpha_l + \alpha_g = 1, \tag{1.2}$$

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whereas ρ_l, ρ_g are densities and u_l, u_g are fluid velocities associated with the liquid and gas phase. Moreover, the three first terms on the right hand side of the momentum equations represent, respectively, wall friction with coefficients f_g, f_l; interfacial friction with coefficient I; and gravity with gravity constant g. Finally, μ_g, μ_l are the viscosity coefficients.

Several challenges are associated with the model (1.1).

- The combination of different density-pressure laws corresponding to the different phases gives rise to non-conventional, nonlinear pressure functions that can be a challenge to deal with;
- Transition to single-phase regions, i.e, regions where m or n become zero, can happen because the volume fractions α_g, α_l become zero and/or because densities ρ_g, ρ_l vanish (formation of vacuum). Typically, we will need some uniform bounds on the masses m and n in order to derive higher order estimates;
- The non-conservative pressure terms $\alpha_g \partial_x P_g$ and $\alpha_l \partial_x P_l$ often prevent from applying arguments used for Navier-Stokes equations;

In [5] a model similar to (1.1) without external force terms is studied for two fluids described by density-pressure relations of the form

$$P_g = C_g \rho_g^{\gamma_g}, \qquad P_l = C_l \rho_l^{\gamma_l}, \qquad \gamma_g, \gamma_l > 1, \qquad C_g, C_l > 0,$$

together with the assumption of equal pressure, i.e., $P_g = P_l = P$. A key assumption in their work is that the viscosity coefficients depend linearly on masses, i.e., are given as

$$\mu_g = \varepsilon_g n, \qquad \mu_l = \varepsilon_l m$$

for positive constants ε_g and ε_l. Thanks to this structure the model allows for deriving a so-called BD entropy estimate, which in turn ensures estimates of the form

$$\int ([m^{1/2}]_x^2 + [n^{1/2}]_x^2) \leq C.$$

From these estimates the well-posedness is obtained as well as information about the long time behavior. We refer also to the recent work [12] which also largely relies on this approach but in a gas-liquid context where a polytropic gas law is used for the gas phase whereas the liquid is assumed to be incompressible.

For the problem with constant viscosity coefficients $\mu_g, \mu_l > 0$ there seems to be very few, if any, mathematical results on compressible two-fluid models, even in one dimension. The purpose of this paper is to provide a first global result for that case. The approach we pursue is to include a *capillary pressure* term, i.e., we do not assume that the two phase pressures P_g and P_l are equal. The assumption about non-equal pressure functions $P_g \neq P_l$ (in an appropriate sense) is quite natural. This amounts to including capillary pressure forces and is commonly included in modeling of two-phase flow i porous media. We refer to [3] for a classical approach and [19] and references therein for a more recent discussion where non-equilibrium effects are taken into considerations.

Capillary pressure P_c is defined as the difference between the non-wetting (nw) fluid and the wetting fluid (w),

$$P_c = P_{nw} - P_w.$$

In a gas-water system gas will be the non-wetting fluid, hence,

$$P_c = P_g - P_l = P_c(\alpha_l),$$

where P_c is decreasing as a function of the wetting phase volume fraction (saturation). Consistent with this we will in our setting assume that

$$P_g - P_l = -f(m), \tag{1.3}$$

where $f'(m) \geq 0$. In this work we shall analyse a reduced version of the full two-fluid model (1.1) where the momentum equations have been simplified. More precisely, we consider the following generic two-fluid hyperbolic-elliptic system based on Stokes equations instead of the full momentum equations:

$$\begin{cases} n_t + (nu_g)_x = 0, \\ m_t + (mu_l)_x = 0, \\ \alpha_g [P_g]_x = \mu_g [u_g]_{xx}, \\ \alpha_l [P_l]_x = \mu_l [u_l]_{xx}, \end{cases} \tag{1.4}$$

where the following constraints are imposed

$$\begin{cases} \alpha_l + \alpha_g = 1, \\ P_l - P_g = f(m), \\ \rho_l = \rho_{l0} + \frac{P_l - P_{l0}}{a_l^2}, \qquad \rho_g = \frac{P_g}{a_g^2}, \end{cases} \tag{1.5}$$

where $f \in C^2([0,\infty))$ and $f' \geq 0$ on $[0,\infty)$. a_l and a_g represent the sound of speed whereas ρ_{l0} represents the density at the reference pressure P_{l0}. These are known parameters. Consider the initial-boundary value conditions:

$$(m, n)(x, 0) = (m_0, n_0)(x) \quad \text{for } x \in [0, 1] \tag{1.6}$$

and

$$(u_l, u_g)(0, t) = (u_l, u_g)(1, t) = 0 \quad \text{for } t \geq 0. \tag{1.7}$$

Finally, we mention that the full two-fluid model equipped with non-equal pressure functions is discussed in the work [13]. We consider the reduced model (1.4) to clearly illustrate the distinct role played by the capillary pressure-like term f which characterizes the difference between the two pressures P_g and P_l.

It is also worth mentioning that the use of Stokes' equations in combination with mass conservation has been used as an alternative approach to the standard description of two-phase flow in porous media based on Darcy's law [16]. In fact there has been quite a lot of work done dealing with this approach as an attempt to take into account certain viscous effects between fluid phases which are ignored in more standard formulations. Hence, the model (1.4) is also of interest from a more applied point of view.

1.1. **Other related works.** The use of non-equal pressure functions $P_g \neq P_l$ in the context of two-fluid modeling is not new. For example, the following inviscid model (no viscosity terms in momentum equations) has been studied in [20] (see also [17]) from a numerical point of view:

$$\begin{aligned} \partial_t(\alpha_g) + u_i \partial_x(\alpha_g) &= q_p(P_g - P_l) \\ \partial_t(n) + \partial_x(nu_g) &= 0 \\ \partial_t(m) + \partial_x(mu_l) &= 0 \\ \partial_t(nu_g) + \partial_x(nu_g^2) + \alpha_g \partial_x P_g + (P_g - P_{ig})\partial_x \alpha_g &= S_g + q_u(u_l - u_g) \\ \partial_t(mu_l) + \partial_x(mu_l^2) + \alpha_l \partial_x P_l + (P_l - P_{il})\partial_x \alpha_l &= S_l. \end{aligned} \tag{1.8}$$

Herein an average interface velocity u_i must be specified as well as some interfacial pressures P_{ig} and P_{il}. Moreover, S_g and S_l are the momentum external force terms whereas q_p (and q_u) is a relaxation source term that will enforce more or less equality between the two pressures P_g and P_l (and u_g and u_l) as q_p (and q_u) becomes sufficiently large. Thus, in this model the use of unequal pressures is compensated for by adding a new equation for the volume fraction α_g.

A similar two-fluid model was discussed in the recent paper [21]. In particular, the construction of solutions of the Riemann problem was explored. The model was written in the form (using the same notation as before)

$$\begin{aligned} \partial_t(\rho_l) + \partial_x(\rho_l u_l) &= 0 \\ \partial_t(n) + \partial_x(n u_g) &= 0 \\ \partial_t(m) + \partial_x(m u_l) &= 0 \\ \partial_t(n u_g) + \partial_x(n u_g^2) + \partial_x(\alpha_g P_g) &= P_g \partial_x \alpha_g \\ \partial_t(m u_l) + \partial_x(m u_l^2) + \partial_x(\alpha_l P_l) &= -P_g \partial_x \alpha_g. \end{aligned} \tag{1.9}$$

The model is derived from a more complete model for gas-solid phase flow, see [2, 6]. Also for this model non-equal pressure functions have been used combined with adding a new PDE equation, in this case, for the solid phase density ρ_l.

Hence, our approach as reflected by relying on the algebraic relation (1.3), which instantly enforces an equilibrium relation between P_g and P_l, seems to be novel and quite different from what has been explored before in the context of inviscid compressible two-fluid modeling.

Finally, before we state the main result of this paper it is also relevant to mention some results on a related gas-liquid model, the so-called drift-flux model. The model is similar to (1.1), however, the two momentum equations have been added together to form a mixture momentum equation. The loss of information about the separate fluid velocities are then compensated for by adding a slip relation of the form $u_g = K[\alpha_g u_g + \alpha_l u_l] + S$ where K and S are known parameters. This algebraic relation is used to obtain a model that has been useful for modeling of realistic gas-liquid flow behavior in different contexts. In particular, the model is general enough to describe important flow behavior where the gas and fluid velocities can be quite different. We refer to [18] and references therein for more information on the model. In [9, 10] this model was analyzed in a setting which involved a free interface separating the gas-liquid mixture from a gas-region whereas in [11] well-posedness was discussed for the model in a closed conduit.

It is interesting to compare the two mentioned two-phase models with other two-phase formulations based on Navier-Stokes equations. A more recent example is represented by the work of Abels and Feireisl [1]. The system consists of the compressible Navier-Stokes equations governing the motion of a mixture of two fluids coupled with the Cahn-Hilliard equation for describing the concentration difference. A general existence result was proved for a 3D model without any restriction on the size of initial data. See also [15] and [8], and references therein, for many interesting results in this direction. This two-phase model involves a common velocity in contrast to the model two-fluid model (1.1) and the drift-flux model studied in [9, 10, 11]. The model has many applications in areas where the two phases do not appear in a mixture, i.e., at one point in space only one of the phases can be present.

1.2. **Main results.** The main result in the paper is concerned about global stability of the solutions near a constant equilibrium state. First, we introduce some notations. Denote $I = [0, 1]$, $\tilde{m} = \int_I m_0\,dx$, $\tilde{n} = \int_I n_0\,dx$. The following (weak) structural assumptions related to the pressure difference $f(m)$ close to 0 and the constant equilibrium state $(\tilde{m}, \tilde{n})$, are needed (see also Remark 2.3):

$$\rho_{l0} - \frac{P_{l0}}{a_l^2} + f(0) > 0, \qquad 0 < f'(\tilde{m}) \le a_l^2, \tag{1.10}$$

where $f \in C^2([0, \infty))$ and $f' \ge 0$. Then, we have the following result.

Theorem 1.1. *Assume that* $m_0 \in H^1$ *where* $\tilde{m} > 0$, $n_0 \in H^1$ *and* $\inf_{x\in[0,1]} n_0 = A_0 > 0$ *and* (1.10) *holds. Then there exists a positive constant* ε_0*, such that the system (1.4) with initial-boundary conditions (1.6) and (1.7) where* $\int_I (m_{0x}^2 + n_{0x}^2) \le \varepsilon_0$*, has a global solution* (m, n, u_l, u_g)*:*
More precisely,

$$(m, n) \in \left[C([0, \infty); H^1)\right]^2 \cap \left[C^1([0, \infty); L^2)\right]^2, \ (u_l, u_g) \in \left[C([0, \infty); H^2)\right]^2$$

and there are constants $N_1, N_2 > 0$ *and* $M_1 > 0$ *(independent of time* t*) such that*

$$0 \le m \le M_1, \qquad 0 < N_1 \le n \le N_2. \tag{1.11}$$

Here the initial data and upper and lower bounds N_1, N_2, and M_1 are related to the constant states $\tilde{m}$ and $\tilde{n}$ by the following inequalities:

$$0 < \tilde{m} \le \sup_I m_0 < M_1, \qquad N_1 < \inf_I n_0 \le \tilde{n} \le \sup_I n_0 < N_2. \tag{1.12}$$

Furthermore, it follows that the solution is globally stable near the constant equilibrium state ($\tilde{m}$, $\tilde{n}$, 0, 0) in the sense that

$$\|(m-\tilde{m}, n-\tilde{n})\|_{H^1} + \|(u_l, u_g)\|_{H^2} \le C\exp\{-Ct\}, \tag{1.13}$$

for all time, where C is a positive constant depending on the initial data and some other known constant but independent of t.

The consequence of this result is that we have a complete description of the long time behavior of (n, m, u_g, u_l) given that the initial masses are sufficiently small in H^1. In particular, in view of the inequality

$$|m_0 - \tilde{m}| \le \int_I |m_{0x}|dx \le \|m_{0x}\|_{L^2} \le \varepsilon_0,$$

and similarly for n_0, it follows that the equilibria $(\tilde{m}, \tilde{n}, 0, 0)$ attracts all solutions emanating from the initial data (m_0, n_0) as long as the distance to the equilibria is sufficiently small and (1.10) and (1.12) holds.

The assumption about non-equal pressure $P_l \neq P_g$ is weak in the sense that we only require that the difference $f(m)$ between P_l and P_g locally around the equilibria state $\tilde{m}$ is increasing, i.e., $f'(\tilde{m}) > 0$.

Outline of our approach. Under the uniform a priori assumptions of m and n as given by (4.58) in Proposition 4.1, we can obtain an estimate of the form $(C_1, C_2, C_3 > 0)$

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_0^1 (C_1[m_x]^2 + C_2[n_x]^2)\Big) + C^*\int_0^1 (C_1[m_x]^2 + C_2[n_x]^2)\\
&\quad \le C_3\Big[\int_0^1 (C_1[m_x]^2 + C_2[n_x]^2)\Big]^{1/2}\Big(\Big[\int_0^1 (C_1[m_x]^2 + C_2[n_x]^2)\Big]^{1/2} + 1\Big)\int_0^1 (C_1[m_x]^2 + C_2[n_x]^2),
\end{aligned}$$

where C^* is a positive constant thanks to the fact (essentially) that $f'(\tilde{m}) > 0$, see (1.10) for the precise statement. It follows that this inequality entails stabilization of m_x and n_x in L^2 under appropriate smallness assumption on initial masses m_0 and n_0 in H^1 in the sense that we can find an $\varepsilon_1 > 0$ such that (Lemma 4.3)

$$\int (m_x^2 + n_x^2) \le \varepsilon_1.$$

Armed with this result we can then close the a priori assumption of Proposition 4.1 which paves the way for proving the existence part of Theorem 1.1 by standard continuity arguments. The asymptotic estimates are a by-product of this analysis as expressed by Lemmas 4.6, 4.7 and 4.8.

The structure of this work is as follows. In Section 2 we provide a local existence result. Then in Section 3, as a preparation for the global estimates we introduce a reformulation of the model that will facilitate the analysis. In Section 4 we give the proof of Theorem 1.1 in terms of Proposition 4.1 (which in turn rely on the results of Lemma 4.2, 4.3, and 4.5). The asymptotic estimates of (m, n, u_l, u_g) follow from Lemma 4.6, 4.7 and 4.8.

2. Local existence

The main objective of this section is to prove a local existence result. For that purpose we will need to control P_{gx} and P_{lx} in some appropriate norms. In particular, it is then necessary with a more precise understanding of how to bound the quantities $\frac{\partial \rho_g}{\partial m}$ and $\frac{\partial \rho_g}{\partial n}$. In the next subsection we provide such insight before we give details of the local existence result in Section 2.2.

2.1. **Relationships between** (ρ_l, ρ_g) **and** (m, n)**.** (1.5) implies that

$$\rho_l = \frac{a_g^2}{a_l^2}\rho_g + \rho_{l0} - \frac{P_{l0}}{a_l^2} + \frac{f(m)}{a_l^2} \tag{2.14}$$

and

$$m\rho_g + n\rho_l = \rho_l\rho_g. \tag{2.15}$$

Substituting (2.14) into (2.15), we have

$$\frac{a_g^2}{a_l^2}\rho_g^2 - b\rho_g - c = 0, \tag{2.16}$$

where

$$\begin{cases} b \doteq b(m,n) = m + \frac{a_g^2}{a_l^2}n + \frac{P_{l0}}{a_l^2} - \rho_{l0} - \frac{f(m)}{a_l^2}, \\ c \doteq c(m,n) = n\Big[\rho_{l0} - \frac{P_{l0}}{a_l^2} + \frac{f(m)}{a_l^2}\Big]. \end{cases} \tag{2.17}$$

From (2.16), we obtain

$$\rho_g = \frac{a_l^2}{2a_g^2}\Big[b \pm \sqrt{b^2 + \frac{4ca_g^2}{a_l^2}}\Big].$$

We need to find some conditions such that we can get a unique, non-negative ρ_g. Since $\rho_{l0} - \frac{P_{l0}}{a_l^2} + f(0) > 0$ and $f' \geq 0$ on $[0, \infty)$, then $c \geq 0$ and thus

$$\begin{cases} \rho_g = \frac{a_l^2}{2a_g^2}\Big[b + \sqrt{b^2 + \frac{4ca_g^2}{a_l^2}}\Big], & \alpha_g = \frac{n}{\rho_g} \\ \rho_l = \frac{a_g^2}{a_l^2}\rho_g + \rho_{l0} - \frac{P_{l0}}{a_l^2} + \frac{f(m)}{a_l^2}, & \alpha_l = \frac{m}{\rho_l}. \end{cases} \tag{2.18}$$

Remark 2.1. *For later use we need some understanding of what is needed to bound* $\frac{\partial \rho_g}{\partial m}$ *and* $\frac{\partial \rho_g}{\partial n}$*. Direct calculations give*

$$\frac{\partial \rho_g(m,n)}{\partial m} = \frac{a_l^2}{2a_g^2}\left(1 - \frac{f'(m)}{a_l^2} + \frac{2b\Big(1 - \frac{f'(m)}{a_l^2}\Big) + \frac{4nf'(m)a_g^2}{a_l^4}}{2\sqrt{b^2 + \frac{4ca_g^2}{a_l^2}}}\right)$$

and

$$\frac{\partial \rho_g(m,n)}{\partial n} = \frac{a_l^2}{2a_g^2}\left(\frac{a_g^2}{a_l^2} + \frac{\frac{2a_g^2 b}{a_l^2} + \frac{4a_g^2\Big[\rho_{l0} - \frac{P_{l0}}{a_l^2} + \frac{f(m)}{a_l^2}\Big]}{a_l^2}}{2\sqrt{b^2 + \frac{4ca_g^2}{a_l^2}}}\right)$$

$$= \frac{1}{2} + \frac{b + 2\Big[\rho_{l0} - \frac{P_{l0}}{a_l^2} + \frac{f(m)}{a_l^2}\Big]}{2\sqrt{b^2 + \frac{4ca_g^2}{a_l^2}}}.$$

In particular, in view of (2.17) *and first part of* (1.10) *we conclude that* $b^2 + 4c\frac{a_g^2}{a_l^2} \geq 4c\frac{a_g^2}{a_l^2}$ *has a positive lower limit if*

$$0 \leq m \leq M, \qquad 0 < \frac{1}{N} \leq n \leq N,$$

for some $M, N > 0$ *from which we also can conclude that* $\left|\frac{\partial \rho_g}{\partial m}\right|$ *and* $\left|\frac{\partial \rho_g}{\partial n}\right|$ *are bounded. Additionally, it also follows that* ρ_g *has a positive lower limit, i.e.,* $\alpha_g = \frac{n}{\rho_g}$ *has an upper bound, and consequently* $0 \leq \frac{n}{\rho_g}, \frac{m}{\rho_l} \leq 1$.

Remark 2.2. *In Lemma 4.3 we will need that* $\tilde{\alpha}_l = \alpha_l(\tilde{m}, \tilde{n}) > 0$ *and* $\tilde{\alpha}_g = \alpha_g(\tilde{m}, \tilde{n}) > 0$*. This follows directly from* (2.18) *and Remark 2.1 since* $\tilde{m}, \tilde{n} > 0$.

Remark 2.3. *In Lemma 4.3 it is crucial that* $\frac{\partial \rho_g(\tilde{m},\tilde{n})}{\partial m} > 0$. *From the assumption that* $1 - f'(\tilde{m})/a_l^2 > 0$ *it follows that*

$$\frac{\partial \rho_g(\tilde{m},\tilde{n})}{\partial m} > \frac{a_l^2}{2a_g^2}\left(\Big[1-\frac{f'(\tilde{m})}{a_l^2}\Big] + \Big[1-\frac{f'(\tilde{m})}{a_l^2}\Big]\frac{b}{\sqrt{b^2+\frac{4ca_g^2}{a_l^2}}}\right) = \frac{a_l^2}{2a_g^2}\Big[1-\frac{f'(\tilde{m})}{a_l^2}\Big]\left(1+\frac{b}{\sqrt{b^2+\frac{4ca_g^2}{a_l^2}}}\right) \geq 0$$

since $c(\tilde{m},\tilde{n}) > 0$, *see* (2.17) *and* (1.10).

2.2. **A local existence result.** The statement of the theorem on local existence of the solution as in Theorem 1.1 is as follows:

Theorem 2.4. *Assume that* $m_0 \in H^1$, $n_0 \in H^1$ *and* $\inf_{x\in[0,1]} n_0 = A_0 > 0$. *Then there exists a positive constant* T_0, *such that the system (1.4) with initial-boundary conditions (1.6) and (1.7) has a solution* (m,n,u_l,u_g) *on* $[0,1]\times[0,T_0]$ *in the sense that*

$$(m,n) \in \big[C([0,T_0];H^1)\big]^2 \cap \big[C^1([0,T_0];L^2)\big]^2,\ (u_l,u_g) \in \big[C([0,T_0];H^2)\big]^2.$$

We are going to apply the iteration arguments to prove Theorem 2.4.

Proof of Theorem 2.4:

Denote

$$S \triangleq S_{T_0,A_1} = \Big\{v \in C([0,T_0];H_0^1\cap H^2)\Big|\|v\|_{C([0,T_0];H^2)} \leq A_\mu A_1\Big\}$$

where $A_\mu = \max\{\frac{1}{\mu_l},\frac{1}{\mu_g}\}$. The constants A_1 and T_0 are specified such that they satisfy (2.33) and (2.48), respectively. Note that C_2 in (2.33) only depends on initial data and known model parameters.

Step 1: construct an iteration sequence.

Following the similar arguments in [7], we construct an iteration sequence as follows:

$$\begin{cases} n_t^k + (n^k u_g^{k-1})_x = 0,\\ m_t^k + (m^k u_l^{k-1})_x = 0,\\ \alpha_g^k[P_g^k]_x = \mu_g[u_g^k]_{xx},\\ \alpha_l^k[P_l^k]_x = \mu_l[u_l^k]_{xx},\ (x,t)\in(0,1)\times(0,T_0], \end{cases} \tag{2.19}$$

with the initial-boundary value conditions:

$$(m^k,n^k)(x,0) = (m_0,n_0)(x) \quad \text{for } x\in[0,1] \tag{2.20}$$

and

$$(u_l^k,u_g^k)(0,t) = (u_l^k,u_g^k)(1,t) = 0 \quad \text{for } t \geq 0, \tag{2.21}$$

for $k = 1,2,3,...$ and $(u_g^0,u_l^0) = (0,0)$, where $\alpha_g^k = \alpha_g(m^k,n^k)$, $\alpha_l^k = \alpha_l(m^k,n^k)$, $P_g^k = P_g(m^k,n^k)$, $P_l^k = P_l(m^k,n^k)$, and

$$(m^k,n^k) \in \big[C([0,T_0];H^1)\big]^2 \cap \big[C^1([0,T_0];L^2)\big]^2,\ (u_g^k,u_l^k) \in \big[C([0,T_0];H_0^1\cap H^2)\big]^2.$$

Moreover, we have

$$m^k \geq \inf_{x\in[0,1]} m_0(x)\exp\left\{-\int_0^{T_0}\|u_{l,x}^{k-1}(s)\|_{L^\infty}ds\right\} \tag{2.22}$$

and

$$n^k \geq \inf_{x\in[0,1]} n_0(x)\exp\{-\int_0^{T_0}\|u_{g,x}^{k-1}(s)\|_{L^\infty}ds\}. \tag{2.23}$$

In the following we will obtain estimates of $\{m^k, n^k, u_l^k, u_g^k\}$ that are independent of k.

Step 2: boundedness of the sequence.

Assume that $u_l^{k-1}, u_g^{k-1} \in S$. To prove $u_l^i, u_g^i \in S$ for all $i = 0, 1, 2, 3, ...$, it suffices to prove $u_l^k, u_g^k \in S$.

In fact, since $u_g^{k-1} \in S$, (2.23) gives

$$n^k \geq A_0 \exp\{-C_1 A_\mu T_0\} \geq A_0 e^{-1} > 0, \tag{2.24}$$

where C_1 is a generic positive constant depending only on A_1 and the initial data, and

$$T_0 \leq \frac{1}{C_1 A_\mu} \triangleq T_1. \tag{2.25}$$

Differentiating $(2.19)_1$ with respect to x, multiplying the result by $2n_x^k$, and integrating by parts over $[0,1]\times[0,t]$ for $t \leq T_0$, we have

$$\begin{aligned}
\int_0^1 |n_x^k|^2 &= \int_0^1 |n_{0,x}|^2 - 3\int_0^t\int_0^1 |n_x^k|^2 u_{g,x}^{k-1} - 2\int_0^t\int_0^1 n^k n_x^k u_{g,xx}^{k-1} \\
&\leq \int_0^1 |n_{0,x}|^2 + C_1 A_\mu \int_0^t\int_0^1 |n_x^k|^2 + 2\int_0^t \|n^k\|_{L^\infty}\left(\int_0^1 |n_x^k|^2\right)^{\frac12}\left(\int_0^1 |u_{g,xx}^{k-1}|^2\right)^{\frac12} \\
&\leq \int_0^1 |n_{0,x}|^2 + C_1 A_\mu T_0 \max_{t\in[0,T_0]}\int_0^1 |n_x^k|^2 + C_1 A_\mu T_0,
\end{aligned} \tag{2.26}$$

where we have used the facts that $u_g^{k-1} \in S$ and that $\displaystyle\int_0^1 n^k = \int_0^1 n_0$, and the Poincaré type inequality

$$\|n^k\|_{L^\infty} \leq \|n^k - \int_0^1 n^k\|_{L^\infty} + \int_0^1 n_0 \leq \|n_x^k\|_{L^2} + \int_0^1 n_0. \tag{2.27}$$

Taking the maximum of both sides of (2.26) over $[0, T_0]$, we have

$$\max_{t\in[0,T_0]}\int_0^1 |n_x^k|^2 \leq 2\int_0^1 n_{0,x}^2 + C_1 A_\mu T_0, \tag{2.28}$$

where we have used

$$T_0 \leq \min\{T_1, \frac{1}{2C_1 A_\mu}\} \triangleq T_2. \tag{2.29}$$

On the other hand, using (2.27) and Hölder inequality, we have

$$\int_0^1 |n^k|^2 \leq \|n^k\|_{L^\infty}^2 \leq \int_0^1 |n_x^k|^2 + \int_0^1 n_0^2. \tag{2.30}$$

(2.28) and (2.30) give

$$\max_{t\in[0,T_0]} \|n^k(\cdot,t)\|_{H^1}^2 \leq 5\|n_0\|_{H^1}^2 + 2C_1 A_\mu T_0 \leq 5\|n_0\|_{H^1}^2 + 1. \tag{2.31}$$

Similarly, we have

$$\max_{t\in[0,T_0]} \|m^k(\cdot,t)\|_{H^1}^2 \leq 5\|m_0\|_{H^1}^2 + 2C_1 A_\mu T_0 \leq 5\|m_0\|_{H^1}^2 + 1, \tag{2.32}$$

provided that $T_0 \leq T_2$.

Using $(2.19)_3$-$(2.19)_4$, the standard elliptic estimates, (2.24), (2.27) (as well as the same kind of estimate for m^k), Remark 2.1, and (2.31)-(2.32), we have

$$\|u_g^k\|_{H^2} \leq C_2 A_\mu \text{ and } \|u_l^k\|_{H^2} \leq C_2 A_\mu,$$

where C_2 depends on A_0 and the initial data, but is independent of A_1 and T_0. Let

$$A_1 \geq C_2, \tag{2.33}$$

we have $u_l^k, u_g^k \in S$ provided that $T_0 \le T_2$. Thus $u_l^i, u_g^i \in S$ for all i provided that $T_0 \le T_2$. Besides, we get (2.31) and (2.32) for all k provided that $T_0 \le T_2$.

Step 3: Compactness arguments.

Since we have $u_l^i, u_g^i \in S$ for all i and (2.31)-(2.32), then there exist a subsequence k_i ($i = 1, 2, 3, ...$) and a (u_l, u_g, m, n), such that

$$\begin{aligned}
&(u_l^{k_i}, u_g^{k_i}) \rightharpoonup (u_l, u_g) \text{ weak-* in } \left[L^\infty([0,T_0]; H_0^1 \cap H^2)\right]^2,\\
&n^{k_i} \rightharpoonup n \text{ weak-* in } L^\infty([0,T_0]; H^1),\\
&m^{k_i} \rightharpoonup m \text{ weak-* in } L^\infty([0,T_0]; H^1),\\
&(n_t^{k_i}, m_t^{k_i}) \rightharpoonup (n_t, m_t) \text{ weak-* in } \left[L^\infty([0,T_0]; L^2)\right]^2
\end{aligned} \tag{2.34}$$

as $k_i \to \infty$, where

$$(u_l, u_g, m, n) \in L^\infty([0,T_0]; H^2) \times L^\infty([0,T_0]; H^2) \times L^\infty([0,T_0]; H^1) \times L^\infty([0,T_0]; H^1),$$

and $n_t, m_t \in L^\infty([0,T_0]; L^2)$. Using the Aubin-Lions's compactness theorem, we can obtain strong convergence. More precisely, there exists a subsequence still denoted by k_i without loss of generality ($i = 1, 2, 3, ...$), such that

$$\begin{aligned}
&n^{k_i} \to n \text{ in } C([0,1] \times [0,T_0]),\\
&m^{k_i} \to m \text{ in } C([0,1] \times [0,T_0]),
\end{aligned} \tag{2.35}$$

as $k_i \to \infty$. $(2.35)_1$ and (2.24) give

$$n(x,t) \ge A_0 e^{-1} > 0 \tag{2.36}$$

for any $(x,t) \in [0,1] \times [0,T_0]$.

Step 4: convergence of $(u_l^{k_i-1}, u_g^{k_i-1})$.

We have shown convergence of $(u_l^{k_i}, u_g^{k_i})$, however, we must also demonstrate convergence of $(u_l^{k_i-1}, u_g^{k_i-1})$ to the same limit functions since both are used in the linearized system (2.19). To that end, denote $\bar{m}^{k+1} = m^{k+1} - m^k$ and $\bar{n}^{k+1} = n^{k+1} - n^k$. Then

$$\begin{cases}
\bar{m}_t^{k+1} + \bar{m}_x^{k+1} u_l^k + m_x^k (u_l^k - u_l^{k-1}) + \bar{m}^{k+1} [u_l^k]_x + m^k (u_l^k - u_l^{k-1})_x = 0,\\
\bar{m}^{k+1}(x,0) = 0
\end{cases} \tag{2.37}$$

and

$$\begin{cases}
\bar{n}_t^{k+1} + \bar{n}_x^{k+1} u_g^k + n_x^k (u_g^k - u_g^{k-1}) + \bar{n}^{k+1} [u_g^k]_x + n^k (u_g^k - u_g^{k-1})_x = 0,\\
\bar{n}^{k+1}(x,0) = 0
\end{cases} \tag{2.38}$$

for $(x,t) \in [0,1] \times [0,T_0]$.

Multiplying (2.37) by $2\bar{m}^{k+1}$, and integrating the result over $[0,1]$, we have

$$\begin{aligned}
\frac{d}{dt}\int_0^1 |\bar{m}^{k+1}|^2 =& -2\int_0^1 \bar{m}^{k+1} m_x^k (u_l^k - u_l^{k-1}) - \int_0^1 |\bar{m}^{k+1}|^2 [u_l^k]_x\\
&-2\int_0^1 \bar{m}^{k+1} m^k (u_l^k - u_l^{k-1})_x\\
\le& 2\|u_l^k - u_l^{k-1}\|_{L^\infty} \|\bar{m}^{k+1}\|_{L^2} \|m_x^k\|_{L^2} + \|[u_l^k]_x\|_{L^\infty} \|\bar{m}^{k+1}\|_{L^2}^2\\
&+ 2\|m^k\|_{L^\infty} \|(u_l^k - u_l^{k-1})_x\|_{L^2} \|\bar{m}^{k+1}\|_{L^2}\\
\le& C_3 \|(u_l^k - u_l^{k-1})_x\|_{L^2} \|\bar{m}^{k+1}\|_{L^2} + C_3 A_\mu \|\bar{m}^{k+1}\|_{L^2}^2,
\end{aligned} \tag{2.39}$$

where C_3 depends only on the initial data and A_1. Here we have used (2.32), the fact that $u_l^k \in S$, and the Poincaré inequality. Similarly, we have

$$\frac{d}{dt}\int_0^1 |\bar{n}^{k+1}|^2 \le C_3\|(u_g^k - u_g^{k-1})_x\|_{L^2}\|\bar{n}^{k+1}\|_{L^2} + C_3A_\mu\|\bar{n}^{k+1}\|_{L^2}^2. \tag{2.40}$$

We see that $u_l^k - u_l^{k-1}$ solves the equation

$$\mu_l[u_l^k - u_l^{k-1}]_{xx} = \alpha_l^k[P_l^k]_x - \alpha_l^{k-1}[P_l^{k-1}]_x = (\alpha_l^k - \alpha_l^{k-1})[P_l^k]_x + \alpha_l^{k-1}[P_l^k - P_l^{k-1}]_x. \tag{2.41}$$

Multiplying (2.41) by $\frac{1}{\mu_l}(u_l^k - u_l^{k-1})$, and integrating by parts over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 |[u_l^k - u_l^{k-1}]_x|^2 \\
=& -\frac{1}{\mu_l}\int_0^1 (u_l^k - u_l^{k-1})(\alpha_l^k - \alpha_l^{k-1})[P_l^k]_x - \frac{1}{\mu_l}\int_0^1 (u_l^k - u_l^{k-1})\alpha_l^{k-1}[P_l^k - P_l^{k-1}]_x \\
=& -\frac{1}{\mu_l}\int_0^1 (u_l^k - u_l^{k-1})(\alpha_l^k - \alpha_l^{k-1})[P_l^k]_x + \frac{1}{\mu_l}\int_0^1 (u_l^k - u_l^{k-1})[\alpha_l^{k-1}]_x(P_l^k - P_l^{k-1}) \\
&+ \frac{1}{\mu_l}\int_0^1 (u_l^k - u_l^{k-1})_x\alpha_l^{k-1}(P_l^k - P_l^{k-1}).
\end{aligned} \tag{2.42}$$

Note that

$$|\alpha_l^k - \alpha_l^{k-1}| \le C_4(|\bar{m}^k| + |\bar{n}^k|) \text{ and } |P_l^k - P_l^{k-1}| \le C_4(|\bar{m}^k| + |\bar{n}^k|)$$

for some positive constant C_4 depending only on A_0 and the initial data. This together with the Poincaré inequality, Hölder inequality and (2.42) implies that

$$\begin{aligned}
\int_0^1 |(u_l^k - u_l^{k-1})_x|^2 \le& \frac{C_4}{\mu_l}\|(u_l^k - u_l^{k-1})_x\|_{L^2}(\|\bar{m}^k\|_{L^2} + \|\bar{n}^k\|_{L^2})\|[P_l^k]_x\|_{L^2} \\
&+ \frac{C_4}{\mu_l}\|(u_l^k - u_l^{k-1})_x\|_{L^2}\|[\alpha_l^{k-1}]_x\|_{L^2}(\|\bar{m}^k\|_{L^2} + \|\bar{n}^k\|_{L^2}) \\
&+ \frac{C_4}{\mu_l}\|(u_l^k - u_l^{k-1})_x\|_{L^2}(\|\bar{m}^k\|_{L^2} + \|\bar{n}^k\|_{L^2}).
\end{aligned} \tag{2.43}$$

Note that

$$P_{lx} = \Big[a_g^2\frac{\partial\rho_g}{\partial m} + f'(m)\Big]m_x + a_g^2\frac{\partial\rho_g}{\partial n}n_x$$

and

$$\alpha_{lx} = -\alpha_{gx} = -\frac{\partial}{\partial x}\frac{n}{\rho_g(m,n)} = \frac{n}{\rho_g^2}\frac{\partial\rho_g}{\partial m}m_x - \Big[\frac{1}{\rho_g} - \frac{n}{\rho_g^2}\frac{\partial\rho_g}{\partial n}\Big]n_x.$$

Since $[P_l^k]_x$ and $[\alpha_l^{k-1}]_x$ are bounded in L^2 due to (2.24), Remark 2.1, (2.31) and (2.32), (2.43) together with the Cauchy inequality gives

$$\int_0^1 |(u_l^k - u_l^{k-1})_x|^2 \le \frac{C_5}{\mu_l^2}(\|\bar{m}^k\|_{L^2}^2 + \|\bar{n}^k\|_{L^2}^2), \tag{2.44}$$

where C_5 is a positive constant depending only on A_0 and the initial data. Similarly, we have

$$\int_0^1 |(u_g^k - u_g^{k-1})_x|^2 \le \frac{C_5}{\mu_g^2}(\|\bar{m}^k\|_{L^2}^2 + \|\bar{n}^k\|_{L^2}^2). \tag{2.45}$$

Combining (2.39), (2.40), (2.44) and (2.45) with Cauchy inequality, we have

$$\frac{d}{dt}\int_0^1 (|\bar{m}^{k+1}|^2 + |\bar{n}^{k+1}|^2) \le C_6A_\mu(\|\bar{m}^k\|_{L^2}^2 + \|\bar{n}^k\|_{L^2}^2) + C_6A_\mu(\|\bar{m}^{k+1}\|_{L^2}^2 + \|\bar{n}^{k+1}\|_{L^2}^2), \tag{2.46}$$

where C_6 is a positive constant depending only on C_3 and C_5. Integrating (2.46) over $[0,t]$ for any given $t\in[0,T_0]$, and taking the maximum on both sides, we have

$$\max_{t\in[0,T_0]}(\|\bar{m}^{k+1}\|_{L^2}^2+\|\bar{n}^{k+1}\|_{L^2}^2)\le\frac{1}{2}\max_{t\in[0,T_0]}(\|\bar{m}^{k}\|_{L^2}^2+\|\bar{n}^{k}\|_{L^2}^2), \tag{2.47}$$

provided

$$T_0\le\min\{T_2,\frac{1}{3C_6A_\mu}\}\triangleq T_3. \tag{2.48}$$

Using (2.47), (2.31) and (2.32), we get

$$\max_{t\in[0,T_0]}(\|\bar{m}^{k+1}\|_{L^2}^2+\|\bar{n}^{k+1}\|_{L^2}^2)\le\left(\frac{1}{2}\right)^{k-1}\max_{t\in[0,T_0]}(\|\bar{m}^{2}\|_{L^2}^2+\|\bar{n}^{2}\|_{L^2}^2)\le C_7\left(\frac{1}{2}\right)^{k-1}, \tag{2.49}$$

for all k, where C_7 is a positive constant depending only on the initial data. (2.44), (2.45) and (2.49) give

$$\|(u_l^k-u_l^{k-1})_x\|_{L^2}^2\le\frac{C_5C_7}{\mu_l^2}\left(\frac{1}{2}\right)^{k-2}, \tag{2.50}$$

and

$$\int_0^1|(u_g^k-u_g^{k-1})_x|^2\le\frac{C_5C_7}{\mu_g^2}\left(\frac{1}{2}\right)^{k-2}. \tag{2.51}$$

(2.50) and (2.51) combined with $(2.34)_1$ imply that

$$(u_l^{k_i-1},u_g^{k_i-1})\rightharpoonup(u_l,u_g)\text{ weak-* in }\left[L^\infty([0,T_0];H_0^1)\right]^2 \tag{2.52}$$

as $k_i\to\infty$.

Step 5: conclusion

Based on (2.34), (2.35) and (2.36) in Step 3, it is easy to verify

$$\begin{aligned}\mu_g[u_g]_{xx}&=\alpha_g[P_g]_x,\\ \mu_l[u_l]_{xx}&=\alpha_l[P_l]_x,\end{aligned} \tag{2.53}$$

a.e. in $[0,1]\times[0,T_0]$, since

$$(\alpha_g^{k_i},\alpha_l^{k_i},P_g^{k_i},P_l^{k_i})\to(\alpha_g,\alpha_l,P_g,P_l)\text{ in }C([0,1]\times[0,T_0]),$$

as $k_i\to\infty$, and $[P_g^{k_i}]_x$ and $[P_l^{k_i}]_x$ are bounded in $L^\infty([0,T_0];L^2)$, where $\alpha_g=\alpha_g(m,n),\alpha_l=\alpha_l(m,n),P_g=P_g(m,n),P_l=P_l(m,n)$.

Using $(2.34)_4$, (2.35), (2.52) and the regularity of (m,n,u_l,u_g), we get

$$\begin{aligned}n_t+(nu_g)_x&=0,\\ m_t+(mu_l)_x&=0,\end{aligned} \tag{2.54}$$

a.e. in $[0,1]\times[0,T_0]$. (2.35) and $(2.34)_1$ ensure that (m,n) and (u_l,u_g) satisfy the initial condition and the boundary condition,respectively. The continuity in time of (m,n,u_l,u_g) can be obtained by using the similar arguments for instance as [7]. Thus, we get a solution (m,n,u_l,u_g) which solves (1.4), (1.6) and (1.7) on $[0,1]\times[0,T_0]$ in the settings as in Theorem 2.4. The uniqueness was done implicitly in Step 4.

3. Reformulation

Recall from Section 1 that $I=[0,1]$, $\tilde{m}=\int_I m_0\,dx$, $\tilde{n}=\int_I n_0\,dx$, $\bar{m}=m-\tilde{m}$ and $\bar{n}=n-\tilde{n}$. Then $(\bar{m},\,\bar{n},\,u_l,\,u_g)$ satisfies

$$\begin{cases}\bar{n}_t+\bar{n}_x u_g+\bar{n}(u_g)_x+\tilde{n}(u_g)_x=0,\\ \bar{m}_t+\bar{m}_x u_l+\bar{m}(u_l)_x+\tilde{m}(u_l)_x=0,\\ a_g^2\alpha_g[\rho_g]_x=\mu_g[u_g]_{xx},\\ a_g^2\alpha_l[\rho_g]_x+\alpha_l f'(m)m_x=\mu_l[u_l]_{xx}.\end{cases}\tag{3.55}$$

Denote $\tilde{\alpha_l}=\alpha_l(\tilde{m},\tilde{n})$, $\tilde{\alpha_g}=\alpha_g(\tilde{m},\tilde{n})$, $\tilde{\rho_l}=\rho_l(\tilde{m},\tilde{n})$, $\tilde{\rho_g}=\rho_g(\tilde{m},\tilde{n})$. Then (3.55) is reformulated as follows:

$$\begin{cases}\bar{n}_t+\bar{n}_x u_g+\bar{n}(u_g)_x+\tilde{n}(u_g)_x=0,\\ \bar{m}_t+\bar{m}_x u_l+\bar{m}(u_l)_x+\tilde{m}(u_l)_x=0,\\ a_g^2(\alpha_g-\tilde{\alpha_g})[\frac{\partial\rho_g}{\partial m}\bar{m}_x+\frac{\partial\rho_g}{\partial n}\bar{n}_x]+a_g^2\tilde{\alpha_g}[\frac{\partial\rho_g}{\partial m}\bar{m}_x+\frac{\partial\rho_g}{\partial n}\bar{n}_x]=\mu_g[u_g]_{xx},\\ a_g^2(\alpha_l-\tilde{\alpha_l})[\frac{\partial\rho_g}{\partial m}\bar{m}_x+\frac{\partial\rho_g}{\partial n}\bar{n}_x]+a_g^2\tilde{\alpha_l}[\frac{\partial\rho_g}{\partial m}\bar{m}_x+\frac{\partial\rho_g}{\partial n}\bar{n}_x]+\alpha_l f'(m)m_x=\mu_l[u_l]_{xx}.\end{cases}\tag{3.56}$$

We will obtain the following refined version:

$$\begin{cases}\bar{n}_t+\tilde{n}(u_g)_x=-\bar{n}_x u_g-\bar{n}(u_g)_x,\\ \bar{m}_t+\tilde{m}(u_l)_x=-\bar{m}_x u_l-\bar{m}(u_l)_x,\\ G_1(m,n,\tilde{m},\tilde{n})\bar{m}_x+G_2(m,n,\tilde{m},\tilde{n})\bar{n}_x+a_g^2\tilde{\alpha_g}[\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\bar{m}_x+\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\bar{n}_x]\\ =\mu_g[u_g]_{xx},\\ G_3(m,n,\tilde{m},\tilde{n})\bar{m}_x+G_4(m,n,\tilde{m},\tilde{n})\bar{n}_x+a_g^2\tilde{\alpha_l}[\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\bar{m}_x+\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\bar{n}_x]\\ +\tilde{\alpha_l}f'(\tilde{m})\bar{m}_x=\mu_l[u_l]_{xx},\end{cases}\tag{3.57}$$

where

$$G_1(m,n,\tilde{m},\tilde{n})=a_g^2(\alpha_g-\tilde{\alpha_g})\frac{\partial\rho_g(m,n)}{\partial m}+a_g^2\tilde{\alpha_g}\left(\frac{\partial\rho_g(m,n)}{\partial m}-\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\right),$$

$$G_2(m,n,\tilde{m},\tilde{n})=a_g^2(\alpha_g-\tilde{\alpha_g})\frac{\partial\rho_g(m,n)}{\partial n}+a_g^2\tilde{\alpha_g}\left(\frac{\partial\rho_g(m,n)}{\partial n}-\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\right),$$

$$G_3(m,n,\tilde{m},\tilde{n})=a_g^2(\alpha_l-\tilde{\alpha_l})\frac{\partial\rho_g(m,n)}{\partial m}+a_g^2\tilde{\alpha_l}\left(\frac{\partial\rho_g(m,n)}{\partial m}-\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\right)+\alpha_l f'(m)-\tilde{\alpha_l}f'(\tilde{m}),$$

and

$$G_4(m,n,\tilde{m},\tilde{n})=a_g^2(\alpha_l-\tilde{\alpha_l})\frac{\partial\rho_g(m,n)}{\partial n}+a_g^2\tilde{\alpha_l}\left(\frac{\partial\rho_g(m,n)}{\partial n}-\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\right).$$

4. Proof of Theorem 1.1

Let C denote a generic constant that is independent of time t but that may depend on some known constants, and T^* denote the maximum time for the existence of solutions as in Theorem 2.4. Theorem 2.4 implies that $T^*>0$. To prove the global existence, it suffices to show that $T^*=\infty$. For otherwise, i.e., $T^*<\infty$, it will lead to a contradiction based on the following estimates.

Proposition 4.1. *Under the conditions of Theorem 1.1, for any $T<T^*$, if*

$$N_1\le n\le N_2 \text{ and } m\le M_1,\tag{4.58}$$

then

$$\bar{N}_1\le n\le \bar{N}_2 \text{ and } m\le \bar{M}_1,\tag{4.59}$$

in $I\times[0,T]$*, provided that* $\int_I (m_{0x}^2+n_{0x}^2)\le \varepsilon_0$ *for some* $\varepsilon_0>0$*. Here* $N_1<\bar N_1<\inf_{x\in I} n_0\le \tilde n$*,* $\tilde n\le \sup_{x\in I} n_0<\bar N_2<N_2$*, and* $\tilde m\le \sup_{x\in I} m_0<\bar M_1<M_1$*.*

The proof of Proposition 4.1 is divided into the following Lemmas 4.2, 4.3 and 4.5.

Lemma 4.2. *Under the assumptions of Proposition 4.1, it holds that*

$$\begin{cases}\mu_g\|[u_g]_x\|_{L^\infty}\le C(\|\bar m\|_{L^\infty}+\|\bar n\|_{L^\infty})(\|m_x\|_{L^1}+\|n_x\|_{L^1}+1),\\ \mu_l\|[u_l]_x\|_{L^\infty}\le C(\|\bar m\|_{L^\infty}+\|\bar n\|_{L^\infty})(\|m_x\|_{L^1}+\|n_x\|_{L^1}+1),\\ \mu_l\|[u_l]_{xx}\|_{L^2}\le C\left(\|m_x\|_{L^2}+\|n_x\|_{L^2}\right),\\ \mu_g\|[u_g]_{xx}\|_{L^2}\le C\left(\|m_x\|_{L^2}+\|n_x\|_{L^2}\right),\end{cases}\tag{4.60}$$

for a.e. $t\in[0,T]$*.*

Proof. Based on $(3.57)_3$, $(3.57)_4$ and the boundary condition, the solutions to $(3.57)_3$ and $(3.57)_4$ can be written respectively as follows

$$\begin{aligned}&\mu_g u_g=\int_0^x\int_0^y G_1(m,n,\tilde m,\tilde n)m_x-x\int_0^1\int_0^y G_1(m,n,\tilde m,\tilde n)m_x+\\&\int_0^x\int_0^y G_2(m,n,\tilde m,\tilde n)n_x-x\int_0^1\int_0^y G_2(m,n,\tilde m,\tilde n)n_x+\\&a_g^2\tilde\alpha_g\int_0^x[\frac{\partial\rho_g(\tilde m,\tilde n)}{\partial m}m+\frac{\partial\rho_g(\tilde m,\tilde n)}{\partial n}n]-a_g^2\tilde\alpha_g x\int_0^1[\frac{\partial\rho_g(\tilde m,\tilde n)}{\partial m}m+\frac{\partial\rho_g(\tilde m,\tilde n)}{\partial n}n]\end{aligned}\tag{4.61}$$

and

$$\begin{aligned}&\mu_l u_l=\int_0^x\int_0^y G_3(m,n,\tilde m,\tilde n)m_x-x\int_0^1\int_0^y G_3(m,n,\tilde m,\tilde n)m_x+\\&\int_0^x\int_0^y G_4(m,n,\tilde m,\tilde n)n_x-x\int_0^1\int_0^y G_4(m,n,\tilde m,\tilde n)n_x+\\&a_g^2\tilde\alpha_l\int_0^x[\frac{\partial\rho_g(\tilde m,\tilde n)}{\partial m}m+\frac{\partial\rho_g(\tilde m,\tilde n)}{\partial n}n]-a_g^2\tilde\alpha_l x\int_0^1[\frac{\partial\rho_g(\tilde m,\tilde n)}{\partial m}m+\frac{\partial\rho_g(\tilde m,\tilde n)}{\partial n}n]+\\&\tilde\alpha_l f'(\tilde m)\int_0^x m-\tilde\alpha_l f'(\tilde m)x\int_0^1 m.\end{aligned}\tag{4.62}$$

Differentiating (4.61) and (4.62) with respect to x, respectively, we have

$$\begin{aligned}\mu_g[u_g]_x=&\int_0^x G_1(m,n,\tilde m,\tilde n)m_x-\int_0^1\int_0^y G_1(m,n,\tilde m,\tilde n)m_x+\\&\int_0^x G_2(m,n,\tilde m,\tilde n)n_x-\int_0^1\int_0^y G_2(m,n,\tilde m,\tilde n)n_x+\\&a_g^2\tilde\alpha_g[\frac{\partial\rho_g(\tilde m,\tilde n)}{\partial m}\bar m+\frac{\partial\rho_g(\tilde m,\tilde n)}{\partial n}\bar n]\end{aligned}\tag{4.63}$$

and

$$\begin{aligned}\mu_l[u_l]_x=&\int_0^x G_3(m,n,\tilde m,\tilde n)m_x-\int_0^1\int_0^y G_3(m,n,\tilde m,\tilde n)m_x+\\&\int_0^x G_4(m,n,\tilde m,\tilde n)n_x-\int_0^1\int_0^y G_4(m,n,\tilde m,\tilde n)n_x+\\&a_g^2\tilde\alpha_l[\frac{\partial\rho_g(\tilde m,\tilde n)}{\partial m}\bar m+\frac{\partial\rho_g(\tilde m,\tilde n)}{\partial n}\bar n]+\tilde\alpha_l f'(\tilde m)\bar m.\end{aligned}\tag{4.64}$$

It is easy to verify that

$$|G_i(m,n,\tilde m,\tilde n)|\le C(|m-\tilde m|+|n-\tilde n|)=C(|\bar m|+|\bar n|)\tag{4.65}$$

for $i = 1, 2, 3, 4$, under the assumption (4.58). Then (4.63) and (4.64) combined with (4.65) deduce $(4.60)_1$ and $(4.60)_2$. From $(3.56)_3$ and $(3.56)_4$, we get $(4.60)_3$ and $(4.60)_4$. □

Lemma 4.3. *Under the assumptions of Proposition 4.1, it holds that*

$$\int_I (m_x^2 + n_x^2) + \int_0^t \int_I (m_x^2 + n_x^2) \le \varepsilon_1 \tag{4.66}$$

for a.e. $t \in [0, T]$, *provided that* $\int_I (m_{0x}^2 + n_{0x}^2) \le \varepsilon_0$, *where* ε_0 *and* ε_1 $(\varepsilon_0 < \varepsilon_1)$ *are independent of* t *and* T.

Proof. Note that $\bar{m}_x = m_x$. We differentiate $(3.57)_2$ with respect to x and get

$$m_{xt} + \tilde{m}(u_l)_{xx} = -m_{xx}u_l - 2m_x(u_l)_x - \bar{m}(u_l)_{xx}. \tag{4.67}$$

Multiplying (4.67) by $\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{\tilde{m}\tilde{\alpha}_l}m_x$, and integrating by parts over I, we have

$$\begin{aligned}
&\frac{d}{dt}\int_I \frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{2\tilde{m}\tilde{\alpha}_l}m_x^2 + \int_I \frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{\tilde{\alpha}_l}m_x(u_l)_{xx} \\
&= -3\int_I \frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{2\tilde{m}\tilde{\alpha}_l}m_x^2(u_l)_x - \int_I \frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{\tilde{m}\tilde{\alpha}_l}\bar{m}m_x(u_l)_{xx}.
\end{aligned} \tag{4.68}$$

Similarly, we have

$$\begin{aligned}
&\frac{d}{dt}\int_I \frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{2\tilde{n}\tilde{\alpha}_g}n_x^2 + \int_I \frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{\tilde{\alpha}_g}n_x(u_g)_{xx} \\
&= -3\int_I \frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{2\tilde{n}\tilde{\alpha}_g}n_x^2(u_g)_x - \int_I \frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{\tilde{n}\tilde{\alpha}_g}\bar{n}n_x(u_g)_{xx}.
\end{aligned} \tag{4.69}$$

On the other hand, we multiply $(3.57)_3$ and $(3.57)_4$ by $\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{\tilde{\alpha}_g}n_x$ and $\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{\tilde{\alpha}_l}m_x$ respectively, and have

$$\begin{aligned}
&\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{\tilde{\alpha}_g}[u_g]_{xx}n_x \\
=&\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{1}{\tilde{\alpha}_g}G_1(m,n,\tilde{m},\tilde{n})m_xn_x + \frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{1}{\tilde{\alpha}_g}G_2(m,n,\tilde{m},\tilde{n})n_x^2 \\
&+ a_g^2[\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}m_x + \frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}n_x]\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}n_x
\end{aligned} \tag{4.70}$$

and

$$\begin{aligned}
&\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{\tilde{\alpha}_l}[u_l]_{xx}m_x \\
=&\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{1}{\tilde{\alpha}_l}G_3(m,n,\tilde{m},\tilde{n})m_x^2 + \frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{1}{\tilde{\alpha}_l}G_4(m,n,\tilde{m},\tilde{n})n_xm_x \\
&+ a_g^2[\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}m_x + \frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}n_x]\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}m_x + \frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}f'(\tilde{m})m_x^2.
\end{aligned} \tag{4.71}$$

(4.70) and (4.71) implies that

$$\begin{aligned}
&a_g^2[\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}m_x + \frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}n_x]\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}n_x \\
=&\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{\tilde{\alpha}_g}[u_g]_{xx}n_x - \frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{1}{\tilde{\alpha}_g}G_1(m,n,\tilde{m},\tilde{n})m_xn_x - \\
&\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{1}{\tilde{\alpha}_g}G_2(m,n,\tilde{m},\tilde{n})n_x^2 \\
\le&\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{\tilde{\alpha}_g}[u_g]_{xx}n_x + C(m_x^2 + n_x^2)(|\bar{m}| + |\bar{n}|)
\end{aligned} \tag{4.72}$$

and

$$\begin{aligned}
&a_g^2[\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}m_x+\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}n_x]\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}m_x+\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}f'(\tilde{m})m_x^2\\
=&\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{\tilde{\alpha_l}}[u_l]_{xx}m_x-\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{1}{\tilde{\alpha_l}}G_3(m,n,\tilde{m},\tilde{n})m_x^2-\\
&\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{1}{\tilde{\alpha_l}}G_4(m,n,\tilde{m},\tilde{n})n_xm_x\\
\leq&\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{\tilde{\alpha_l}}[u_l]_{xx}m_x+C(m_x^2+n_x^2)(|\bar{m}|+|\bar{n}|),
\end{aligned}\tag{4.73}$$

where we have used Cauchy inequality and (4.65). Summing (4.72) and (4.73) on both sides, we have

$$\begin{aligned}
&a_g^2[\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}m_x+\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}n_x]^2+\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}f'(\tilde{m})m_x^2\\
\leq&\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{\tilde{\alpha_g}}[u_g]_{xx}n_x+\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{\tilde{\alpha_l}}[u_l]_{xx}m_x+C(m_x^2+n_x^2)(|\bar{m}|+|\bar{n}|).
\end{aligned}\tag{4.74}$$

Summing (4.68) and (4.69) results in

$$\begin{aligned}
&\frac{d}{dt}\int_I\left[\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{2\tilde{m}\tilde{\alpha_l}}m_x^2+\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{2\tilde{n}\tilde{\alpha_g}}n_x^2\right]+\int_I\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{\tilde{\alpha_l}}m_x(u_l)_{xx}+\\
&\int_I\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{\tilde{\alpha_g}}n_x(u_g)_{xx}\\
=&-3\int_I\left[\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{2\tilde{m}\tilde{\alpha_l}}m_x^2(u_l)_x+\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{2\tilde{n}\tilde{\alpha_g}}n_x^2(u_g)_x\right]-\\
&\int_I\left[\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{\tilde{m}\tilde{\alpha_l}}\bar{m}m_x(u_l)_{xx}+\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{\tilde{n}\tilde{\alpha_g}}\bar{n}n_x(u_g)_{xx}\right].
\end{aligned}\tag{4.75}$$

Substituting (4.60) to (4.75), we have

$$\begin{aligned}
&\frac{d}{dt}\int_I\left[\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{2\tilde{m}\tilde{\alpha_l}}m_x^2+\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{2\tilde{n}\tilde{\alpha_g}}n_x^2\right]+\int_I\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{\tilde{\alpha_l}}m_x(u_l)_{xx}+\\
&\int_I\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{\tilde{\alpha_g}}n_x(u_g)_{xx}\\
\leq&C\mu_l\|(u_l)_x\|_{L^\infty}\int_I m_x^2+C\mu_g\|(u_g)_x\|_{L^\infty}\int_I n_x^2+\\
&C\mu_l\|\bar{m}\|_{L^\infty}\int_I\left(m_x^2+(u_l)_{xx}^2\right)+C\mu_g\|\bar{n}\|_{L^\infty}\int_I\left(n_x^2+(u_g)_{xx}^2\right)\\
\leq&C(\|\bar{m}\|_{L^\infty}+\|\bar{n}\|_{L^\infty})(\|m_x\|_{L^1}+\|n_x\|_{L^1}+1)\int_I\left(m_x^2+n_x^2\right).
\end{aligned}\tag{4.76}$$

Integrating (4.74) with respect to x over I, and adding the result to (4.76), we get

$$\begin{aligned}
&\frac{d}{dt}\int_I\left[\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}\frac{\mu_l}{2\tilde{m}\tilde{\alpha_l}}m_x^2+\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}\frac{\mu_g}{2\tilde{n}\tilde{\alpha_g}}n_x^2\right]+a_g^2\int_I[\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}m_x+\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial n}n_x]^2\\
&+\frac{\partial\rho_g(\tilde{m},\tilde{n})}{\partial m}f'(\tilde{m})\int_I m_x^2\leq C(\|\bar{m}\|_{L^\infty}+\|\bar{n}\|_{L^\infty})(\|m_x\|_{L^1}+\|n_x\|_{L^1}+1)\int_I\left(m_x^2+n_x^2\right).
\end{aligned}$$

Then there exist positive constants $\bar{A}_i$ for $i=0,1,2,3,4,5,6$ such that

$$\begin{aligned}
&\frac{d}{dt}\int_I(\bar{A}_0m_x^2+\bar{A}_1n_x^2)+\bar{A}_2\int_I(\bar{A}_3m_x+\bar{A}_4n_x)^2+\bar{A}_5\int_I m_x^2\\
&\leq\bar{A}_6(\|\bar{m}\|_{L^\infty}+\|\bar{n}\|_{L^\infty})(\|m_x\|_{L^1}+\|n_x\|_{L^1}+1)\int_I\left(\bar{A}_0m_x^2+\bar{A}_1n_x^2\right).
\end{aligned}\tag{4.77}$$

Note that here we make use of the assumption (1.10) which guarantees that $\bar{A}_5 > 0$. See also Remark 4.4. Then, in view of the basic inequality $a^2 + b^2 \le 2(a+b)^2 + 3b^2$ there exists a positive constant $\bar{A}_7$ such that

$$\begin{aligned}&\frac{d}{dt}\int_I (\bar{A}_0 m_x^2 + \bar{A}_1 n_x^2) + \bar{A}_7 \int_I (\bar{A}_0 m_x^2 + \bar{A}_1 n_x^2)\\ &\le \bar{A}_6(\|\bar{m}\|_{L^\infty} + \|\bar{n}\|_{L^\infty})(\|m_x\|_{L^1} + \|n_x\|_{L^1} + 1)\int_I \left(\bar{A}_0 m_x^2 + \bar{A}_1 n_x^2\right).\end{aligned} \tag{4.78}$$

By the Poincaré inequality, we have

$$\|\bar{m}\|_{L^\infty} \le C\|m_x\|_{L^1} \tag{4.79}$$

and

$$\|\bar{n}\|_{L^\infty} \le C\|n_x\|_{L^1}. \tag{4.80}$$

(4.78) combined with (4.79) and (4.80) turns out that there exists a positive constant A_8 such that

$$\begin{aligned}&\frac{d}{dt}\int_I (\bar{A}_0 m_x^2 + \bar{A}_1 n_x^2) + \bar{A}_7 \int_I (\bar{A}_0 m_x^2 + \bar{A}_1 n_x^2)\\ &\le \bar{A}_8(\|m_x\|_{L^1} + \|n_x\|_{L^1})(\|m_x\|_{L^1} + \|n_x\|_{L^1} + 1)\int_I \left(\bar{A}_0 m_x^2 + \bar{A}_1 n_x^2\right)\\ &\le \bar{A}_8\Big[\int_I \left(\bar{A}_0 m_x^2 + \bar{A}_1 n_x^2\right)\Big]^{1/2}\Big(\Big[\int_I \left(\bar{A}_0 m_x^2 + \bar{A}_1 n_x^2\right)\Big]^{1/2} + 1\Big)\int_I \left(\bar{A}_0 m_x^2 + \bar{A}_1 n_x^2\right),\end{aligned} \tag{4.81}$$

where $\bar{A}_8$ is redefined in the transition from the second to the third line. (4.81) corresponds to an ODI of the form

$$\frac{d}{dt}g + g \le \sqrt{g}(\sqrt{g} + 1)g, \qquad t > 0, \qquad g \sim \int_I (\bar{A}_0 m_x^2 + \bar{A}_1 n_x^2).$$

Together with standard continuity arguments this ODI implies that there exist positive constants ε_0 and ε_1 ($\varepsilon_0 < \varepsilon_1$) depending only on $\bar{A}_0, \bar{A}_1, \bar{A}_7$ and $\bar{A}_8$ such that if $\int_I (m_{0x}^2 + n_{0x}^2) \le \varepsilon_0$, then $\int_I (m_x^2 + n_x^2) \le \varepsilon_1$ and

$$\frac{d}{dt}\int_I (\bar{A}_0 m_x^2 + \bar{A}_1 n_x^2) + \frac{\bar{A}_7}{2}\int_I (\bar{A}_0 m_x^2 + \bar{A}_1 n_x^2) \le 0 \tag{4.82}$$

for a.e. $t \in [0, T]$.

□

Remark 4.4. *For an equal-pressure two-fluid model $P_l = P_g$ and $f(m) = 0$. Consequently, the "$\bar{A}_5$ term" in (4.77) disappears and it seems that we cannot obtain the crucial ODI (4.78) that enforces stability.*

Lemma 4.5. *Under the assumptions of Proposition 4.1, it holds that*

$$\bar{N}_1 \le n \le \bar{N}_2 \text{ and } m \le \bar{M}_1 \tag{4.83}$$

for all $(x, t) \in I \times [0, T]$.

Proof. (4.79) and (4.80) give

$$m \le \tilde{m} + C\|m_x\|_{L^1} \tag{4.84}$$

and

$$\tilde{n} - C\|n_x\|_{L^1} \le n \le \tilde{n} + C\|n_x\|_{L^1}. \tag{4.85}$$

Since we have $\bar{M}_1 > \tilde{m}$ and $\bar{N}_1 < \tilde{n} < \bar{N}_2$, and (4.82) implies that ε_1 can be taken sufficiently small if ε_0 is small, thus we let ε_0 small enough, such that (4.83) can be concluded by (4.84) and (4.85). □

By this, the proof of Proposition 4.1 is complete. By Proposition 4.1 and the standard continuity arguments, we get (4.58) for all $(x,t) \in I \times [0,T^*)$ as well as the conclusions in Lemmas 4.2 and (4.3) for a.e. $(x,t) \in I \times [0,T^*)$. These imply that $T^* = \infty$. Thus, we finish the proof of global existence. The uniqueness can be done by using arguments similar to those used in Step 4 of Section 2.

Now we are in a position to prove the time-decay rate (1.13). More precisely, we have

Lemma 4.6. *Under the conditions of Theorem 1.1, for a.e. $t \in [0,\infty)$, it holds that*

$$\int_I (m_x^2 + n_x^2) \le C \exp\{-Ct\}, \tag{4.86}$$

where C depends on $\bar{A}_0, \bar{A}_1$ and $\bar{A}_7$ but is independent of t and T.

Proof. (4.86) can be obtained easily by using (4.82) since the ODI $\frac{dg}{dt} + \lambda g \le 0$ implies that $(e^{\lambda t} g)' \le 0$. □

Lemma 4.7. *Under the conditions of Theorem 1.1, for a.e. $t \in [0,\infty)$, it holds that*

$$\int_I (\bar{m}^2 + \bar{n}^2) \le C \exp\{-Ct\}, \tag{4.87}$$

where C depends on $\bar{A}_0, \bar{A}_1$ and $\bar{A}_7$ but is independent of t and T.

Proof. Using Poincaré inequality, we have

$$\int_I (\bar{m}^2 + \bar{n}^2) \le \int_I (m_x^2 + n_x^2). \tag{4.88}$$

(4.88) and (4.86) give us the (4.87). □

Lemma 4.8. *Under the conditions of Theorem 1.1, for a.e. $t \in [0,\infty)$, it holds that*

$$\|(u_l, u_g)\|_{H^2}^2 \le C \exp\{-Ct\} \tag{4.89}$$

where C depends on $\bar{A}_0, \bar{A}_1$ and $\bar{A}_7$ but is independent of t and T.

Proof. (4.89) can be obtained by using $(4.60)_3$, $(4.60)_4$, and (4.86). □

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