ANALYSIS OF A COMPRESSIBLE TWO-FLUID STOKES SYSTEM WITH CONSTANT VISCOSITY

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Abstract. Basic properties of a reduced viscous compressible gas-liquid two-fluid model are explored. The model is composed of two conservation laws representing mass balance for gas and liquid coupled to two elliptic equations (Stokes system) for the two fluid velocities and obtained by ignoring acceleration terms in the full momentum equations. First, we present a result that shows existence and uniqueness of regular solutions for a fixed time $T_0 > 0$ which depends on the initial data and the constant viscosity coefficients. Moreover, T_0 can be large when the viscosity coefficients are large. However, for a fixed set of viscosity coefficients, we conjecture that the smooth solution might blow up, at least, as time tends to infinity. This result is backed up by considering a numerical example for a fixed set of viscosity coefficients demonstrating that for smooth and small initial data with no single-phase regions, the solution may tend to produce both single-phase regions and blow-up of mass gradients as time becomes large.

Keyword: two-fluid model, Navier-Stokes, wellbore flow systems, cell dynamics, existence, uniqueness, blow-up
Subject classification: 76T10, 76N10, 65M12, 35L60

1. Introduction

Model formulation. We are interested in studying a 1D transient two-phase model for isentropic gas-liquid flow relevant for a wide range of applications. The model is based on the so-called two-fluid formulation where the gas and liquid phase have separate mass and momentum conservation equations. In particular, the momentum equations involve a non-conservative pressure-related term, a viscous term and external force terms representing gravity and friction between fluid and wall as well as interfacial friction. The model takes the following form [17] (Chapter 10):

$$\begin{aligned} &\partial_t(n) + \partial_x(nu_g) = 0 \\ &\partial_t(m) + \partial_x(mu_l) = 0 \\ &\partial_t(nu_g) + \partial_x(nu_g^2) + \alpha_g\partial_x P = -f_g u_g - C(u_g - u_l) - ng + \partial_x(\mu_g \partial_x u_g) \\ &\partial_t(mu_l) + \partial_x(mu_l^2) + \alpha_l\partial_x P = -f_l u_l + C(u_g - u_l) - mg + \partial_x(\mu_l \partial_x u_l). \end{aligned} \tag{1}$$

Here $n = \alpha_g\rho_g$ and $m = \alpha_l\rho_l$ where the volume fractions satisfy

$$\alpha_l + \alpha_g = 1, \tag{2}$$

ρ_l, ρ_g are densities and u_l, u_g are fluid velocities associated with the liquid and gas phase, respectively.

This kind of two-fluid gas-liquid model plays a crucial role for the industry involved in the construction of new and safe wellbore flow systems that may operate under extreme conditions (high pressure and high temperature). When the model is used to study deepwater wellbore operations there are many challenging phenomena that can occur. Some of them are: (i) dynamic transition zones separating two-phase and single-phase regions; (ii) strong expansion effects related to compressed gas which moves upwardly towards a lower pressure; (iii) complicated friction terms to take into account more realistic flow patterns; (iv) transition from one flow regime to another; (v)

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fluid flow between the wellbore and surrounding reservoir. A good understanding of mathematical properties of (1) is important both for increased insight into physical mechanisms that will dictate the behavior of the flow system as well as for the construction of reliable discrete versions of (1).

From a mathematical point of view several inherent challenges are associated with the model. When the velocities are assumed to be equal, i.e., $u_l = u_g$, by summing the two momentum equations in (1) and using (2), we obtain a mixture momentum equation where the coefficient of P_x becomes 1. In this case, there have been several works about (global) well posedness of weak and regular solutions as well as the long-time behaviors, refer for instance to [8, 9, 10, 11, 13, 15, 19, 20, 21]. See [7] for more on the relation between those two different type of models.

When the velocities are unequal new challenges arise, such as the fact that pressure terms are not in conservative form. As a consequence, some essential estimates seem more difficult to obtain for general viscosity coefficients. More recently, some elegant works have been published on the global well-posedness of solutions of the model (1)-(2) without considering the gravity, the friction and the interfacial friction. Please refer to [2, 3] for global weak solutions, where capillarity effects are considered in [2] in a three-dimensional settings. The arguments in [2, 3] rely particularly on having specific mass-dependent viscosities corresponding to $\mu_l = m$ and $\mu_g = n$. For constant viscosity coefficients or more general mass-dependent viscosity laws, it still seems unknown whether the global weak solutions exist or not.

Our aim is to consider the two-phase Stokes system (obtained by ignoring acceleration terms in the momentum equations of (1)) with constant viscosity coefficients and establish its well-posedness in a fixed-time interval. Additionally, we would also like to gain some understanding of the solutions as time becomes large by carrying out a relevant numerical experiment. Although the system is a simplified version, most likely it can still capture some of the main properties of the full model (1).

More specifically, we will consider a polytropic gas law and a weakly compressible liquid represented by

$$\rho_g = \frac{P}{a_g^2}, \qquad \rho_l = \rho_{l,0} + \frac{P - p_0}{a_l^2} \qquad (k_0 \triangleq \rho_{l,0} - \frac{p_0}{a_l^2} > 0). \tag{3}$$

Consequently, from the relation $\frac{m}{\rho_l} + \frac{n}{\rho_g} = 1$ we get

$$P = P(m,n) = a_l^2(\rho_l - \rho_{l,0}) + p_0 = a_g^2 \rho_g = \frac{a_l^2}{2}\Big[b + \sqrt{b^2 + \frac{4ca_g^2}{a_l^2}}\Big], \tag{4}$$

where

$$\begin{cases} b \doteq b(m,n) = m + \frac{a_g^2}{a_l^2} n - k_0, \\ c \doteq c(m,n) = k_0 n. \end{cases} \tag{5}$$

As already mentioned, in this work we will focus on a reduced version of (1) where inertial effects in the momentum equations have been neglected and the viscosity coefficients μ_l, μ_g are positive constants. The model then takes the form

$$\begin{aligned} \partial_t(n) + \partial_x(nu_g) &= 0 \\ \partial_t(m) + \partial_x(mu_l) &= 0 \\ \alpha_g \partial_x P &= -ng + \mu_g \partial_{xx}^2 u_g \\ \alpha_l \partial_x P &= -mg + \mu_l \partial_{xx}^2 u_l, \end{aligned} \tag{6}$$

where g is a positive constant (gravity constant). The model is equipped with the boundary data

$$u_i(x=0,t) = u_i(x=1,t) = 0, \qquad i = l, g. \tag{7}$$

and initial data

$$m(x,t=0) = m_0(x), \qquad n(x,t=0) = n_0(x), \qquad x \in [0,1], \tag{8}$$

for $i = l, g$. As will be demonstrated numerically in Section 4 this model is general enough to qualitatively describe important gas-liquid dynamics like segregation of gas and liquid in a vertical conduit involving a dynamic transition zone between mixture and pure phase regions.

Novelty of this work. A model similar to (1), however, with mass-dependent viscosity terms and no external force terms, was studied in [3]. The authors also assumed that both phases have pressure-density relations of the form $P = C\rho^{\gamma}$ with constants $C > 0$ and $\gamma > 1$ (polytropic gas law), which is a different situation compared to the gas-liquid setting described by (3). It was demonstrated in [3] that masses m and n are in $W^{1,2}$ for all times thanks to the possibility to derive a so-called BD-type estimate. The analysis presented here for the reduced model (6) demonstrate that m and n are in $W^{1,r}$ ($r \geq 2$) (i.e., they remain continuous) for a fixed time $T_0 > 0$ which depends on the initial data and the viscosity coefficients. Consequently, we cannot claim that m and n are in $W^{1,r}$ for all times for a fixed set of viscosity coefficients. In fact, this result is backed up by a numerical example demonstrating that gradients in m and n might blow up as time becomes large due to a natural separation of heavy liquid and light gas in a vertical closed conduit. This indicates that the model we study may contain phenomena not covered by the model studied in [3]. Note also that there are additional challenges with the pressure law (4) as explained in Remark 2.4.

Relevance of the model (6) for other applications. Before we describe precisely our result it can be interesting to say something more about the potential relevance of the two-phase Stokes model (6) beyond fluid mechanic related problems [6, 18]. Description of the dynamics of two phases that move with individual fluid velocities can be done by formulating conservation of mass equations for each phase combined with separate momentum equations. For example, in modeling of cell dynamics, a new interpretation of the well-known Keller-Segel model was presented in [4] based on a two-phase formulation. More precisely, using the above notation a model of the following form was presented

$$\begin{aligned} \partial_t(\alpha_c) + \partial_x(\alpha_c u_c) &= S_c \\ \partial_t(\alpha_w) + \partial_x(\alpha_w u_w) &= -S_c, \qquad \alpha_c + \alpha_w = 1, \\ \alpha_c \partial_x P + \partial_x(\alpha_c \Lambda) &= -C(u_c - u_w) + \partial_x(\mu_c \partial_x u_c) \\ \alpha_w \partial_x P &= +C(u_c - u_w) + \partial_x(\mu_w \partial_x u_w). \end{aligned} \tag{9}$$

The two phases involved here are cells (c) and water (w) which are assumed to be incompressible fluids. However, in addition to these transport equations for cells and water, there is another transport-reaction model for a chemical component which strongly affects the dynamics of the cells. In particular, there is a "new" term Λ (potential function) in the momentum equation for cells that can take into account how the cells' behavior differs from that of the bulk water due to its sensitivity to the chemical component. There is also typically a source term S_c in the mass balance equation. In conclusion, the model (9) clearly has its own specialities but the Stokes type two-fluid model (6) might be considered as a submodel of it. Hence, insight into fundamental properties of (6) seems relevant for the understanding of the model (9).

The rest of the paper is organized as follows: In Section 2 we give a precise statement of the result we achieve for (6). In Section 3, we prove Theorem 2.1 by using iteration arguments (due to a linearization of the mass balance equations) combined with the particle equations and energy estimates. Then, in Section 4, for a fixed set of viscosity coefficients, we give a numerical example demonstrating that gradients in m and n might blow up as time becomes large. Finally, we give some concluding remarks in Section 5.

2. Result

In the rest of the paper, we denote $W^{k,p} = W^{k,p}([0,1])$ and $L^p = L^p([0,1])$ for $k = 1, 2$ and $p \in [1, \infty]$.

Assumptions. The following assumptions about initial masses m_0 and n_0 are considered. See Remark 2.4 for more explanation.

(i).

$$m_0, n_0 \in W^{1,r},\ k_0 < C_1 \le m_0 \le C_2,\ 0 \le n_0 \le C_3,\ \text{in } [0,1], \tag{10}$$

for any given constant $r \ge 2$, where C_i is a positive constant for $i = 1, 2, 3$.

(ii).

$$m_0, n_0 \in W^{1,r},\ 0 \le m_0 \le C_4 < k_0,\ 0 \le n_0 \le C_5,\ \text{in } [0,1], \tag{11}$$

for any given constant $r \ge 2$, where C_i is a positive constant for $i = 4, 5$.

(iii).

$$m_0, n_0 \in W^{1,r},\ 0 \le m_0 \le C_6, 0 < C_7 \le n_0 \le C_8,\ \text{in } [0,1], \tag{12}$$

for any given constant $r \ge 2$, where C_i is a positive constant for $i = 6, 7, 8$.

Main result. We now present a precise statement of a result that gives some insight into the role played by the viscosity coefficients μ_g and μ_l versus the regularity (smoothness) of the solution. A main message is that the mass variables m and n remain continuous (since m_x and n_x are in L^r for $r \ge 2$) as long as the viscosity is large enough for the specified time interval $[0, T_0]$ (or equivalently, the time T_0 is small enough for a given set of viscosity coefficients). In order to shed some more light on this conclusion we will then carry out a numerical experiment in Section 4 whose result seems to fit well with the theoretical result because it demonstrates that if we do not put restriction on the length of the time interval represented by T_0, the mass variables m and n apparently create a sharp transition zone corresponding to a separation of heavy liquid at the bottom and light gas at the top of the flow domain. In other words, we may not control that m, n are in $W^{1,r}$.

Theorem 2.1. *Under one of the assumptions (10), (11) or (12), there exists a positive constant T_0 which satisfies (53) and a solution (n, m, u_g, u_l) of (6)–(8) on $[0,1] \times [0, T_0]$ in the sense that*

$$(m, n) \in C([0, T_0]; W^{1,r}), \qquad (m_t, n_t) \in C([0, T_0]; L^r), \qquad (u_l, u_g) \in C([0, T_0]; W^{2,r}). \tag{13}$$

Moreover, upper-lower bounds of m and n hold as specified in (40), (41) and (42).

Remark 2.1. *A closer inspection of the condition* (53) *on the existence time T_0 reveals that it can be chosen large if $A_\mu = \max\{\frac{1}{\mu_l}, \frac{1}{\mu_g}\}$ is sufficiently small, i.e., viscosities μ_l and μ_g are sufficiently large.*

Remark 2.2. *One can find in the proof of Theorem 2.1 that we only need that $r > 1$ in each step, except step 4. In order to deal properly with the non-conservative pressure terms $\alpha_g P_x$ and $\alpha_l P_x$ we will then in fact need that $r \ge 2$.*

Remark 2.3. *Note that $\alpha_l = \alpha_l(m, n) = \frac{m}{\rho_l(m,n)}$ is well-defined as $\rho_l = \rho_l(P) \ge \rho_{l,0} - \frac{p_0}{a_l^2} = k_0 > 0$ for $m, n \ge 0$. Hence, if $0 \le m \le M$ for some positive M it follows that $\alpha_l = \frac{m}{\rho_l} \le \frac{M}{k_0} < \infty$. From $\frac{m}{\rho_l} + \frac{n}{\rho_g} = 1$ it follows that $0 \le \frac{m}{\rho_l}, \frac{n}{\rho_g} \le 1$.*

Remark 2.4. *Direct calculations show that*

$$\frac{\partial \rho_g(m,n)}{\partial m} = \frac{a_l^2}{2a_g^2}\left(1 + \frac{b}{\sqrt{b^2 + \frac{4ca_g^2}{a_l^2}}}\right), \qquad \frac{\partial \rho_g(m,n)}{\partial n} = \frac{1}{2} + \frac{b + 2k_0}{2\sqrt{b^2 + \frac{4ca_g^2}{a_l^2}}}$$

where

$$b^2 + \frac{4ca_g^2}{a_l^2} = (m - k_0)^2 + 2mn\frac{a_g^2}{a_l^2} + \frac{a_g^4}{a_l^4}n^2 + 2k_0\frac{a_g^2}{a_l^2}n.$$

The different assumptions on the upper-lower bounds of m_0 and n_0 in (10)-(12) are used to guarantee that $|\frac{\partial P}{\partial n}| \sim |\frac{\partial \rho_g}{\partial n}|$ is bounded (note that the bound on $|\frac{\partial P}{\partial m}|$ is always ensured) subject to the condition that it can be shown that m and n satisfy bounds of the same kind. This is in fact *obtained in (28)–(30) for the approximate system (14). Consequently, it follows that P_x can be controlled by m_x and n_x which is used repeatedly in Section 2.*

3. Proof of Theorem 2.1

Denote

$$S \triangleq S_{T_0,A_1} = \Big\{ v \in C([0,T_0]; W_0^{1,r} \cap W^{2,r}) \Big| \|v\|_{C([0,T_0];W^{2,1})} \le A_\mu A_1 \Big\}$$

where $r \ge 2$, $A_\mu = \max\{\frac{1}{\mu_l}, \frac{1}{\mu_g}\}$, A_1 and T_0 satisfy (36) and (53), respectively. Note that the constant C in (36) only depends on initial data and other known model parameters.

Step 1: construction of an iteration sequence.
Following the similar arguments in [5], we construct an approximate system through an iteration sequence as follows:

$$\begin{cases} n_t^k + (n^k u_g^{k-1})_x = 0, \\ m_t^k + (m^k u_l^{k-1})_x = 0, \\ \alpha_g^k [P^k]_x = -n^k g + \mu_g [u_g^k]_{xx}, \\ \alpha_l^k [P^k]_x = -m^k g + \mu_l [u_l^k]_{xx}, \ (x,t) \in (0,1) \times (0, T_0], \end{cases} \tag{14}$$

with the initial-boundary value conditions:

$$(m^k, n^k)(x,0) = (m_0, n_0)(x) \quad \text{for } x \in [0,1] \tag{15}$$

and

$$(u_l^k, u_g^k)(0,t) = (u_l^k, u_g^k)(1,t) = 0 \quad \text{for } t \ge 0, \tag{16}$$

for $k = 1,2,3,...$, where $(u_g^0, u_l^0) = (0,0)$, $\alpha_g^k = \alpha_g(m^k, n^k)$, $\alpha_l^k = \alpha_l(m^k, n^k)$, $P^k = P(m^k, n^k)$, and

$$(m^k, n^k) \in C([0,T_0]; W^{1,r}) \cap C^1([0,T_0]; L^r), \ (u_g^k, u_l^k) \in C([0,T_0]; W_0^{1,r} \cap W^{2,r}).$$

Step 2: boundedness of the sequence.
By virtue of $(14)_1$, we have

$$n^k(x,t) = n_0(X_g^k(0;x,t)) \exp\{-\int_0^t u_{g,X_g^k}^{k-1}(X_g^k(\tau;x,t),\tau)\, d\tau\}, \tag{17}$$

where X_g^k is a solution of the particle equation:

$$\begin{cases} \frac{dX_g^k(\tau;x,t)}{d\tau} = u_g^{k-1}(X_g^k(\tau;x,t),\tau), \ \tau \in (0,T_0], \\ X_g^k(t;x,t) = x \end{cases} \tag{18}$$

for each k, where $(x,t) \in [0,1] \times [0,T_0]$. Lemma 1.2 in [14] tells us that the unique solution $X_g^k(\tau;x,t)$ of (18) and its partial derivatives with respect to τ, x and t are continuous on $[0,T_0] \times [0,1] \times [0,T_0]$. Moreover,

$$\frac{\partial X_g^k(\tau;x,t)}{\partial x} = \exp\{\int_t^\tau u_{g,X_g^k}^{k-1}(X_g^k(s;x,t),s)\, ds\}. \tag{19}$$

As a consequence of (17) and (19), we have

$$\inf_{x\in[0,1]} n_0(x) \exp\{-\int_0^{T_0} \|u_{g,x}^{k-1}(s)\|_{L^\infty} ds\} \le n^k \le \sup_{x\in[0,1]} n_0(x) \exp\{\int_0^{T_0} \|u_{g,x}^{k-1}(s)\|_{L^\infty} ds\} \tag{20}$$

and

$$\begin{aligned} \frac{\partial n^k(x,t)}{\partial x} =& \frac{dn_0(X_g^k(0;x,t))}{dX_g^k} \exp\{-2\int_0^t u_{g,X_g^k}^{k-1}(X_g^k(\tau;x,t),\tau)\, d\tau\} - \\ & n^k(x,t) \int_0^t u_{g,X_g^k X_g^k}^{k-1}(X_g^k(\tau;x,t),\tau) \frac{\partial X_g^k(\tau;x,t)}{\partial x}\, d\tau. \end{aligned} \tag{21}$$

Similarly, we have

$$\inf_{x\in[0,1]} m_0(x)\exp\left\{-\int_0^{T_0}\|u_{l,x}^{k-1}(s)\|_{L^\infty}ds\right\}\le m^k\le \sup_{x\in[0,1]} m_0(x)\exp\left\{\int_0^{T_0}\|u_{l,x}^{k-1}(s)\|_{L^\infty}ds\right\} \tag{22}$$

and

$$\begin{aligned}\frac{\partial m^k(x,t)}{\partial x} =&\frac{dm_0(X_l^k(0;x,t))}{dX_l^k}\exp\{-2\int_0^t u_{l,X_l^k}^{k-1}(X_l^k(\tau;x,t),\tau)\,d\tau\}-\\ &m^k(x,t)\int_0^t u_{l,X_l^kX_l^k}^{k-1}(X_l^k(\tau;x,t),\tau)\frac{\partial X_l^k(\tau;x,t)}{\partial x}\,d\tau,\end{aligned} \tag{23}$$

where X_l^k is a unique solution of (18) with u_g^{k-1} replaced by u_l^{k-1}.

Assume that $u_l^{k-1}, u_g^{k-1}\in S$. To prove $u_l^i, u_g^i\in S$ for all $i=0,1,2,3,...$, it suffices to prove $u_l^k, u_g^k\in S$.

In fact, as a consequence of that $u_g^{k-1}, u_l^{k-1}\in S$, we have

$$\|u_{g,x}^{k-1}(\cdot,t)\|_{L^\infty}\le\|u_{g,x}^{k-1}(\cdot,t)\|_{W^{1,1}}\le A_\mu A_1,\ \text{and}\ \|u_{l,x}^{k-1}(\cdot,t)\|_{L^\infty}\le\|u_{l,x}^{k-1}(\cdot,t)\|_{W^{1,1}}\le A_\mu A_1 \tag{24}$$

for $t\in[0,T_0]$.

Under the assumption (i), we have

$C_1\exp\{-A_\mu A_1T_0\}\le m^k\le C_2\exp\{A_\mu A_1T_0\},$ and $0\le n^k\le C_3\exp\{A_\mu A_1T_0\}$ on $[0,1]\times[0,T_0]$.

where we have used (20), (22) and (24). In this case, we let

$$T_0\le\frac{1}{A_\mu A_1}\log\big(\frac{C_1}{\bar C_1}\big)\triangleq T_1 \text{ for } \bar C_1\in(k_0,C_1). \tag{25}$$

Under the assumption (ii), we have

$$0\le m^k\le C_4\exp\{A_\mu A_1T_0\},\ \text{and}\ 0\le n^k\le C_5\exp\{A_\mu A_1T_0\}\text{ on }[0,1]\times[0,T_0].$$

In this case, we let

$$T_0\le\frac{1}{A_\mu A_1}\log\big(\frac{\bar C_4}{C_4}\big)\triangleq \bar T_1 \text{ for } \bar C_4\in(C_4,k_0). \tag{26}$$

Under the assumption (iii), we have

$0\le m^k\le C_6\exp\{A_\mu A_1T_0\}$, and $C_7\exp\{-A_\mu A_1T_0\}\le n^k\le C_8\exp\{A_\mu A_1T_0\}$ on $[0,1]\times[0,T_0]$.

where we have used (20), (22) and (24). In this case, we let

$$T_0\le\frac{1}{A_\mu A_1}\triangleq \bar{\bar T}_1. \tag{27}$$

Consequently, we have

- Under the assumption (i), we have
$$k_0<\bar C_1\le m^k\le\frac{C_1C_2}{\bar C_1},\ \text{and}\ 0\le n^k\le\frac{C_1C_3}{\bar C_1}\text{ on }[0,1]\times[0,T_0], \tag{28}$$
where $T_0\le T_1$.
- Under the assumption (ii), we have
$$0\le m^k\le\bar C_4<k_0,\ \text{and}\ 0\le n^k\le\frac{\bar C_4C_5}{C_4}\quad\text{on }[0,1]\times[0,T_0], \tag{29}$$
where $T_0\le\bar T_1$.
- Under the assumption (iii), we have
$$0\le m^k\le C_6e,\ \text{and}\ \frac{C_7}{e}\le n^k\le C_8e\qquad\text{on }[0,1]\times[0,T_0], \tag{30}$$
where $T_0\le\bar{\bar T}_1$.

It follows from (28), (29), (30) and Remark 2.4 that under the assumption (10), (11) or (12), there exists a positive constant C which is independent of T_0, A_μ and A_1, such that

$$\begin{cases} \|\frac{\partial P(m^k,n^k)}{\partial n^k}\|_{L^\infty([0,1]\times[0,T_0])} \leq C, \ \|\frac{\partial P(m^k,n^k)}{\partial m^k}\|_{L^\infty([0,1]\times[0,T_0])} \leq C, \\ \|m^k\|_{L^\infty([0,1]\times[0,T_0])} \leq C, \ \|n^k\|_{L^\infty([0,1]\times[0,T_0])} \leq C, \end{cases} \tag{31}$$

where T_0 satisfies (25), (26) or (27).

Now we are in a position to prove $u_l^k, u_g^k \in S$. First, we derive L^1 estimates of n_x^k and m_x^k. By virtue of (19) and (21), we have

$$\begin{aligned} \int_0^1 |\frac{\partial n^k(x,t)}{\partial x}|dx \leq & \int_0^1 |\frac{dn_0(X_g^k(0;x,t))}{dX_g^k} \exp\{-2\int_0^t u^{k-1}_{g,X_g^k}(X_g^k(\tau;x,t),\tau)\,d\tau\}|\,dx + \\ & \int_0^1 |n^k(x,t)\int_0^t u^{k-1}_{g,X_g^kX_g^k}(X_g^k(\tau;x,t),\tau)\frac{\partial X_g^k(\tau;x,t)}{\partial x}\,d\tau|\,dx \\ \leq & \int_0^1 |\frac{dn_0(x)}{dx}|\exp\{\int_0^t \|u^{k-1}_{g,x}(\cdot,\tau)\|_{L^\infty}d\tau\}\,dx + C\int_0^t\int_0^1 |u^{k-1}_{g,xx}(x,\tau)|\,dx\,d\tau \\ \leq & \exp\{A_\mu A_1 T_0\}\|n_{0,x}\|_{L^1} + CA_\mu A_1 T_0, \end{aligned} \tag{32}$$

where we have used the fact that $u_g^{k-1} \in S$. Similarly, we have

$$\int_0^1 |\frac{\partial m^k(x,t)}{\partial x}|dx \leq \exp\{A_\mu A_1 T_0\}\|m_{0,x}\|_{L^1} + CA_\mu A_1 T_0. \tag{33}$$

By virtue of the elliptic equation $(14)_3$, the solution u_g^k can be expressed by

$$u_g^k = \frac{1}{\mu_g}\int_0^x\int_0^y (\alpha_g^k[P^k]_\xi + n^k g)\,d\xi\,dy - \frac{1}{\mu_g}x\int_0^1\int_0^y (\alpha_g^k[P^k]_\xi + n^k g)\,d\xi\,dy$$

which implies that

$$[u_g^k]_x = \frac{1}{\mu_g}\int_0^x (\alpha_g^k[P^k]_\xi + n^k g)\,d\xi - \frac{1}{\mu_g}\int_0^1\int_0^y (\alpha_g^k[P^k]_\xi + n^k g)\,d\xi\,dy.$$

Consequently,

$$\|u_g^k(\cdot,t)\|_{W^{2,1}} \leq \frac{5}{\mu_g}\|(\alpha_g^k[P^k]_x + n^k g)\|_{L^1} \leq 5A_\mu\big[C(\|m_x^k\|_{L^1} + \|n_x^k\|_{L^1}) + g\|n_0\|_{L^1}\big] \tag{34}$$

for all $t \in [0,T_0]$, where we have used the fact that $\int_0^1 n^k = \int_0^1 n_0$. Similarly, we have

$$\|u_l^k(\cdot,t)\|_{W^{2,1}} \leq 5A_\mu\big[C(\|m_x^k\|_{L^1} + \|n_x^k\|_{L^1}) + g\|m_0\|_{L^1}\big]$$

for all $t \in [0,T_0]$. Denote $T_2 = \frac{1}{A_\mu A_1}$ and let

$$\begin{aligned} & T_0 \leq \min\{T_1, T_2\} && \text{for assumption (10)}, \\ & T_0 \leq \min\{\bar{T}_1, T_2\} && \text{for assumption (11)}, \\ & T_0 \leq \min\{\bar{\bar{T}}_1, T_2\} && \text{for assumption (12)}, \end{aligned} \tag{35}$$

and

$$A_1 \geq 5Ce(\|m_{0,x}\|_{L^1} + \|n_{0,x}\|_{L^1}) + 10C^2 + 5g\|m_0\|_{L^1} + 5g\|n_0\|_{L^1}. \tag{36}$$

Then for assumptions (10), (11) and (12) respectively, we have

$$\|u_g^k(\cdot,t)\|_{W^{2,1}} \leq A_\mu A_1 \text{ and } \|u_l^k(\cdot,t)\|_{W^{2,1}} \leq A_\mu A_1$$

for all $t \in [0,T_0]$. Consequently, we have $u_l^k, u_g^k \in S$. It also follows from (32) and (33) that

$$\|n_x^k(\cdot,t)\|_{L^1} \leq C, \qquad \|m_x^k(\cdot,t)\|_{L^1} \leq C \tag{37}$$

for an appropriate choice of C independent of A_μ, A_1, and T_0.

Step 3: Compactness arguments.
In order to obtain some compactness properties of m_x^k and n_x^k, we need to bound them in L^r where $r > 1$[1]. In fact, since we have $u_l^i, u_g^i \in S$ for all i and have (31), (32) and (33), we evaluate $\|m^k(\cdot,t)\|_{W^{1,r}}$, $\|n^k(\cdot,t)\|_{W^{1,r}}$ and $\|u_l^k(\cdot,t)\|_{W^{2,r}}$, $\|u_g^k(\cdot,t)\|_{W^{2,r}}$ as we did in (32) and (34) respectively. This gives inequalities of the form

$$\int_0^1 |[u_g^{k-1}]_{xx}|^r \le C_\mu \int_0^1 ([m^{k-1}]_x^r + [n^{k-1}]_x^r) + C_\mu \qquad \text{(similarly for } u_l^{k-1}\text{)}$$

and

$$\int_0^1 ([m^k]_x^r + [n^k]_x^r) \le C_\mu \int_0^t \int_0^1 ([m^{k-1}]_x^r + [n^{k-1}]_x^r) + C_\mu.$$

Then we use Gronwall inequality and verify that m^k and n^k are bounded in $C([0,T_0];W^{1,r})$ uniformly for k and that u_l^k and u_g^k are bounded in $C([0,T_0];W^{2,r})$ uniformly for k.

Thus, there exist a subsequence k_i $(i=1,2,3,...)$ and a (u_l,u_g,m,n), such that

$$\begin{aligned} &(u_l^{k_i}, u_g^{k_i}) \rightharpoonup (u_l,u_g) \text{ weak-* in } L^\infty([0,T_0];W_0^{1,r}\cap W^{2,r}),\\ &n^{k_i} \rightharpoonup n \text{ weak-* in } L^\infty([0,T_0];W^{1,r}),\\ &m^{k_i} \rightharpoonup m \text{ weak-* in } L^\infty([0,T_0];W^{1,r}),\\ &(n_t^{k_i}, m_t^{k_i}) \rightharpoonup (n_t,m_t) \text{ weak-* in } L^\infty([0,T_0];L^r) \end{aligned} \tag{38}$$

as $k_i \to \infty$, where

$$(u_l,u_g) \in L^\infty([0,T_0];W^{2,r}\cap W_0^{1,r}), \text{ and } (m,n) \in L^\infty([0,T_0];W^{1,r}),$$

and $n_t, m_t \in L^\infty([0,T_0];L^r)$. Using the Aubin-Lions' compactness theorem, we can obtain some strong convergence. More precisely, there exists a subsequence still denoted by k_i without loss of generality $(i=1,2,3,...)$, such that

$$\begin{aligned} &n^{k_i} \to n \text{ in } C([0,1]\times[0,T_0]),\\ &m^{k_i} \to m \text{ in } C([0,1]\times[0,T_0]), \end{aligned} \tag{39}$$

as $k_i \to \infty$. It follows from (39), (28), (29) and (30) that

- Under the assumption (i), we have
$$k_0 < \bar{C}_1 \le m \le \frac{C_1C_2}{\bar{C}_1}, \text{ and } 0 \le n \le \frac{C_1C_3}{\bar{C}_1} \text{ on } [0,1]\times[0,T_0], \tag{40}$$
where $T_0 \le \min\{T_1,T_2\}$.
- Under the assumption (ii), we have
$$0 \le m \le \bar{C}_4 < k_0, \text{ and } 0 \le n \le \frac{\bar{C}_4C_5}{C_4} \quad \text{on } [0,1]\times[0,T_0], \tag{41}$$
where $T_0 \le \min\{\bar{T}_1,T_2\}$.
- Under the assumption (iii), we have
$$0 \le m \le C_6 e, \text{ and } \frac{C_7}{e} \le n \le C_8 e \qquad \text{on } [0,1]\times[0,T_0], \tag{42}$$
where $T_0 \le \min\{\bar{\bar{T}}_1,T_2\}$.

Step 4: convergence of $(u_l^{k_i-1}, u_g^{k_i-1})$.

We are going to investigate the convergence of the neighbor sequence of $(u_l^{k_i}, u_g^{k_i})$, i.e., $(u_l^{k_i-1}, u_g^{k_i-1})$, in order to make sure that their limits are the same, since they both appear in the approximate system (14).

[1]In the paper, we need $r \ge 2$ due to the non-conservative form of the momentum equations. Please refer to (48)-(50) in Step 4.

To that end, we need the estimates of the difference between m^{k+1} (n^{k+1}) and m^k (n^k), since there is a connection between velocity and mass due to the momentum equation. Denote $\bar{m}^{k+1} = m^{k+1} - m^k$ and $\bar{n}^{k+1} = n^{k+1} - n^k$. Then

$$\begin{cases} \bar{m}_t^{k+1} + \bar{m}_x^{k+1} u_l^k + m_x^k (u_l^k - u_l^{k-1}) + \bar{m}^{k+1}[u_l^k]_x + m^k (u_l^k - u_l^{k-1})_x = 0, \\ \bar{m}^{k+1}(x,0) = 0 \end{cases} \tag{43}$$

and

$$\begin{cases} \bar{n}_t^{k+1} + \bar{n}_x^{k+1} u_g^k + n_x^k (u_g^k - u_g^{k-1}) + \bar{n}^{k+1}[u_g^k]_x + n^k (u_g^k - u_g^{k-1})_x = 0, \\ \bar{n}^{k+1}(x,0) = 0 \end{cases} \tag{44}$$

for $(x,t) \in [0,1] \times [0,T_0]$.

Using (43), we have

$$\begin{aligned} \frac{d}{dt}\int_0^1 |\bar{m}^{k+1}|^r \leq & r\int_0^1 |\bar{m}^{k+1}|^{r-1}|m_x^k||u_l^k - u_l^{k-1}| + (r-1)\int_0^1 |\bar{m}^{k+1}|^r |[u_l^k]_x| \\ & + r\int_0^1 |\bar{m}^{k+1}|^{r-1}|m^k||(u_l^k - u_l^{k-1})_x| \\ \leq & r\|u_l^k - u_l^{k-1}\|_{L^\infty}\|\bar{m}^{k+1}\|_{L^r}^{r-1}\|m_x^k\|_{L^r} + (r-1)\|[u_l^k]_x\|_{L^\infty}\|\bar{m}^{k+1}\|_{L^r}^r \\ & + r\|m^k\|_{L^\infty}\|(u_l^k - u_l^{k-1})_x\|_{L^r}\|\bar{m}^{k+1}\|_{L^r}^{r-1} \\ \leq & \bar{C}\|(u_l^k - u_l^{k-1})_x\|_{L^r}\|\bar{m}^{k+1}\|_{L^r}^{r-1} + \bar{C}A_\mu\|\bar{m}^{k+1}\|_{L^r}^r, \end{aligned} \tag{45}$$

where $\bar{C}$ is a generic positive constant depending only on the initial data, the upper bound of T_0 and other known constants but independent of k and A_μ. Here we have used the facts that m_x^k is bounded in L^r and that $u_l^k \in S$, and the Poincaré inequality. Similarly, we have

$$\frac{d}{dt}\int_0^1 |\bar{n}^{k+1}|^r \leq \bar{C}\|(u_g^k - u_g^{k-1})_x\|_{L^r}\|\bar{n}^{k+1}\|_{L^r}^{r-1} + \bar{C}A_\mu\|\bar{n}^{k+1}\|_{L^r}^r. \tag{46}$$

It is easy to see that $u_l^k - u_l^{k-1}$ solves the equation

$$\begin{aligned} \mu_l[u_l^k - u_l^{k-1}]_{xx} = & \alpha_l^k P_x^k - \alpha_l^{k-1}P_x^{k-1} + g(m^k - m^{k-1}) \\ = & (\alpha_l^k - \alpha_l^{k-1})P_x^k + \alpha_l^{k-1}[P^k - P^{k-1}]_x + g(m^k - m^{k-1}) \\ = & (\alpha_l^k - \alpha_l^{k-1})P_x^k + (\alpha_l^{k-1}[P^k - P^{k-1}])_x - \alpha_{lx}^{k-1}[P^k - P^{k-1}] + g(m^k - m^{k-1}). \end{aligned} \tag{47}$$

Similar to the estimate of u_g^k (see (34)), $\|(u_l^k - u_l^{k-1})_x\|_{L^r}$ can be evaluated as follows:

$$\begin{aligned} \|[u_l^k - u_l^{k-1}]_x\|_{L^r} \leq & \frac{\bar{C}}{\mu_l}\|(\alpha_l^k - \alpha_l^{k-1})P_x^k\|_{L^1} + \frac{\bar{C}}{\mu_l}\|[\alpha_l^{k-1}]_x(P^k - P^{k-1})\|_{L^1} + \\ & \frac{\bar{C}}{\mu_l}\|\alpha_l^{k-1}(P^k - P^{k-1})\|_{L^r} + \frac{\bar{C}}{\mu_l}\|\bar{m}^k\|_{L^r}. \end{aligned} \tag{48}$$

Note that

$$|\alpha_l^k - \alpha_l^{k-1}| \leq \bar{C}(|\bar{m}^k| + |\bar{n}^k|) \text{ and } |P^k - P^{k-1}| \leq \bar{C}(|\bar{m}^k| + |\bar{n}^k|).$$

Since

$$\alpha_{lx} = \frac{\partial}{\partial x}\frac{m}{\rho_l(m,n)} = \Big[\frac{1}{\rho_l} - \frac{m}{\rho_l^2}\frac{1}{a_l^2}\frac{\partial P}{\partial m}\Big]m_x - \frac{m}{\rho_l^2}\frac{1}{a_l^2}\frac{\partial P}{\partial n}n_x,$$

it follows, in view of Remark 2.3 and (31), that

$$|[\alpha_l^{k-1}]_x| \leq \bar{C}(|m_x^{k-1}| + |n_x^{k-1}|).$$

Similarly, for P_x^k, and we can conclude that P_x^k and $[\alpha_l^{k-1}]_x$ are in L^r. Using Hölder inequality we can estimate the first term on the RHS of (48) as follows

$$\|(\alpha_l^k - \alpha_l^{k-1})P_x^k\|_{L^1}^r \leq \bar{C}\int_0^1 (|\bar{m}^k| + |\bar{n}^k|)^r \cdot \Big(\int_0^1 |P_x^k|^{\frac{r}{r-1}}\Big)^{r-1} \leq \bar{C}\int_0^1 (|\bar{m}^k|^r + |\bar{n}^k|^r),$$

if $\frac{r}{r-1} \leq r$, i.e., $r \geq 2$. The same arguments give that

$$\|[\alpha_l^{k-1}]_x(P^k - P^{k-1})\|_{L^1}^r \leq \bar{C}\int_0^1 (|\bar{m}^k|^r + |\bar{n}^k|^r),$$

while it follows that

$$\|\alpha_l^{k-1}(P^k - P^{k-1})\|_{L^r}^r \leq \bar{C}\int_0^1 (|\bar{m}^k|^r + |\bar{n}^k|^r).$$

Consequently,

$$\|[u_l^k - u_l^{k-1}]_x\|_{L^r} \leq \bar{C}A_\mu(\|\bar{m}^k\|_{L^r} + \|\bar{n}^k\|_{L^r}). \tag{49}$$

Similarly, we have

$$\|[u_g^k - u_g^{k-1}]_x\|_{L^r} \leq \bar{C}A_\mu(\|\bar{m}^k\|_{L^r} + \|\bar{n}^k\|_{L^r}). \tag{50}$$

Combining (45), (46), (49) and (50) with Young inequality, we have

$$\frac{d}{dt}\int_0^1 (|\bar{m}^{k+1}|^r + |\bar{n}^{k+1}|^r) \leq \bar{C}A_\mu(\|\bar{m}^k\|_{L^r}^r + \|\bar{n}^k\|_{L^r}^r) + \bar{C}A_\mu(\|\bar{m}^{k+1}\|_{L^r}^r + \|\bar{n}^{k+1}\|_{L^r}^r). \tag{51}$$

Integrating (51) over $[0,t]$ for any given $t \in [0,T_0]$, and taking the maximum on both sides, we have

$$\max_{t\in[0,T_0]}(\|\bar{m}^{k+1}\|_{L^r}^r + \|\bar{n}^{k+1}\|_{L^r}^r) \leq \frac{1}{2}\max_{t\in[0,T_0]}(\|\bar{m}^k\|_{L^r}^r + \|\bar{n}^k\|_{L^r}^r), \tag{52}$$

provided

$$\begin{aligned} T_0 &\leq \min\{T_1, T_2, \frac{1}{3\bar{C}A_\mu}\} \qquad \text{for assumption (10)},\\ T_0 &\leq \min\{\bar{T}_1, T_2, \frac{1}{3\bar{C}A_\mu}\} \qquad \text{for assumption (11)},\\ T_0 &\leq \min\{\bar{\bar{T}}_1, T_2, \frac{1}{3\bar{C}A_\mu}\} \qquad \text{for assumption (12)}. \end{aligned} \tag{53}$$

(52) implies that

$$\max_{t\in[0,T_0]}(\|\bar{m}^{k+1}\|_{L^r}^r + \|\bar{n}^{k+1}\|_{L^r}^r) \leq \left(\frac{1}{2}\right)^{k-1}\max_{t\in[0,T_0]}(\|\bar{m}^2\|_{L^r}^r + \|\bar{n}^2\|_{L^r}^r) \leq \bar{C}\left(\frac{1}{2}\right)^{k-1}, \tag{54}$$

for all k. (49), (50) and (54) give

$$\|(u_l^k - u_l^{k-1})_x\|_{L^r}^r \leq \bar{C}A_\mu^r\left(\frac{1}{2}\right)^{k-2}, \tag{55}$$

and

$$\|(u_g^k - u_g^{k-1})_x\|_{L^r}^r \leq \bar{C}A_\mu^r\left(\frac{1}{2}\right)^{k-2}. \tag{56}$$

(55) and (56) combined with $(38)_1$ imply that

$$(u_l^{k_i-1}, u_g^{k_i-1}) \rightharpoonup (u_l, u_g) \text{ weak-* in } L^\infty([0,T_0]; W_0^{1,r}) \tag{57}$$

as $k_i \to \infty$.

Step 5: conclusion

Based on (38), (39), (40), (41) and (42) in Step 3, it is easy to verify

$$\begin{aligned} \mu_g[u_g]_{xx} &= ng + \alpha_g P_x,\\ \mu_l[u_l]_{xx} &= mg + \alpha_l P_x, \end{aligned} \tag{58}$$

a.e. in $[0,1]\times[0,T_0]$, since

$$(\alpha_g^{k_i}, \alpha_l^{k_i}, P^{k_i}) \to (\alpha_g, \alpha_l, P) \text{ in } C([0,1]\times[0,T_0]),$$

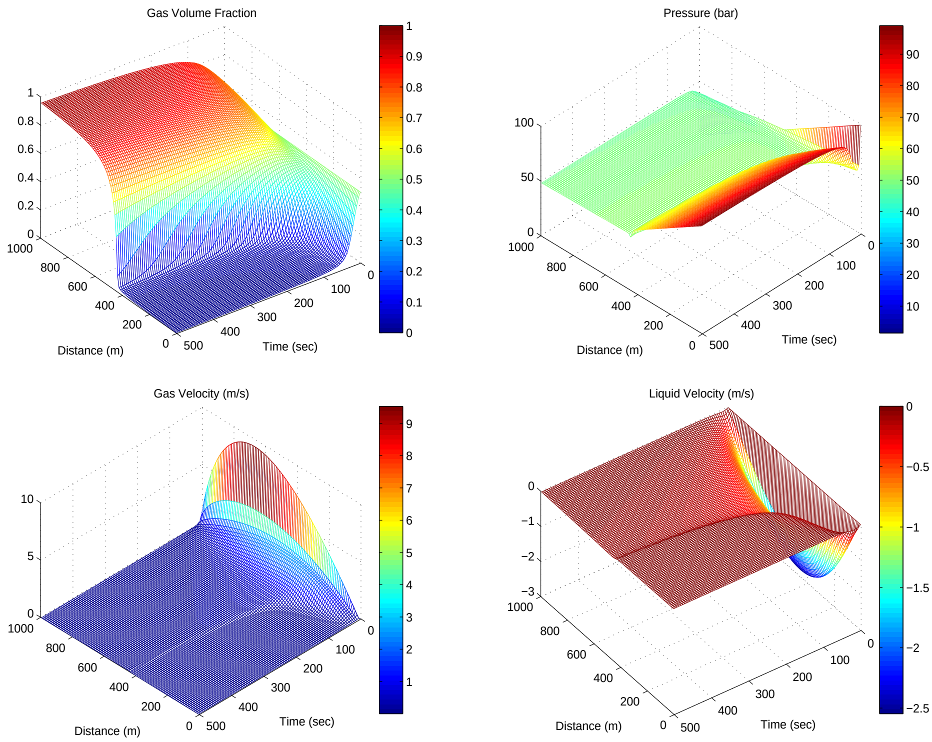

FIGURE 1. An example of the behavior of the model (6) showing how initial smooth initial data with no single-phase zones tend to develop single-phase regions and sharp gradients in the volume fraction.

as $k_i \to \infty$, and $P_x^{k_i}$ is bounded in $L^\infty([0,T_0];L^2)$, where $\alpha_g = \alpha_g(m,n), \alpha_l = \alpha_l(m,n), P = P(m,n)$.

Using $(38)_4$, (39), (57) and the regularity of (m,n,u_l,u_g), we get

$$\begin{aligned} n_t + (nu_g)_x &= 0, \\ m_t + (mu_l)_x &= 0, \end{aligned} \tag{59}$$

a.e. in $[0,1]\times[0,T_0]$. (39) and $(38)_1$ ensure that (m,n) and (u_l,u_g) satisfy the initial condition and the boundary condition, respectively. The continuity in time of (m,n) can be proved by using the characteristic equation as in Step 2. Then we use the equations of (u_l,u_g) and get the continuity in time of (u_l,u_g). Thus, we get a solution (m,n,u_l,u_g) which solves (6), (7) and (8) on $[0,1]\times[0,T_0]$ in the settings as in Theorem 2.1. The uniqueness was done implicitly in Step 4.

4. A NUMERICAL EXAMPLE

We explore an example for the model (6) subject to the following initial data

$$\alpha_g(x,t=0) = 0.5, \qquad P(x,t=0) = P_0(x).$$

From these we can compute $\rho_l(P_0)$ and $\rho_g(P_0)$, then the corresponding $m_0(x)$, $n_0(x)$ and $u_{l,0}(x)$, $u_{g,0}(x)$. We consider a fully discrete version of the approximate system (14) and compute solutions on a spatial grid corresponding to 100 cells. Plots of α_g, P, $\alpha_l u_l$, and $\alpha_g u_g$ are are shown in Fig. 1. The plots tell us that the heavy liquid will start to fall towards the bottom of the vertical conduit (giving rise to a negative liquid velocity) whereas the light gas is displaced upwardly (giving rise to a positive gas velocity). As more and more of the two phases have been separated, the fluids will gradually stop moving reflected by the dying fluid velocities. Associated with this separation there is formation of a sharp gradient in the volume fraction representing an interface between the two phases.

5. Concluding remarks

The analysis has given some insight into important mechanisms of the reduced two-fluid model (6). It is demonstrated that for a fixed time $T > 0$, there exists a strong solution as described by Theorem 2.1 subject to the condition that the viscosity coefficients are sufficiently large. In particular, it is shown that masses m and n are in $W^{1,2}$. The fact that this result depends on time T is quite natural, as reflected by the numerical example, since is is demonstrated that the gradient of the volume fraction tend to blow up for larger times indicating that m and n may not remain in $W^{1,2}$ for all times. An interesting open question is whether blow-up of gradients in masses might happen in finite time for fixed viscosity coefficients for the model (6).

The formulation of the two-fluid model itself is not cast in stone. Future studies will focus on seeking appropriate formulations of the two-fluid model and related functional spaces in which to study such models where results (estimates) can be obtained with weaker conditions on the viscosity coefficients.

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