GLOBAL WEAK SOLUTIONS FOR A COMPRESSIBLE GAS-LIQUID MODEL WITH WELL-FORMATION INTERACTION

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Abstract. The objective of this work is to explore a compressible gas-liquid model designed for modeling of well flow processes. We build into the model well-reservoir interaction by allowing flow of gas between well and formation (surrounding reservoir). Inflow of gas and subsequent expansion of gas as it ascends towards the top of the well (a so-called gas kick) represents a major concern for various well operations in the context of petroleum engineering. We obtain a global existence result under suitable assumptions on the regularity of initial data and the rate function that controls the flow of gas between well and formation. Uniqueness is also obtained by imposing more regularity on the initial data. The key estimates are to obtain appropriate lower and upper bounds on the gas and liquid masses. For that purpose we introduce a transformed version of the original model that is highly convenient for analysis of the original model. In particular, in the analysis of the transformed model additional terms, representing well-formation interaction, can be treated by natural extensions of arguments that previously have been employed for the single-phase Navier-Stokes model. The analysis ensures that transition to single-phase regions do not appear when the initial state is a true gas-liquid mixture.

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Key words. two-phase flow, well-reservoir flow, weak solutions, Lagrangian coordinates, free boundary problem

1. Introduction

Many well operations in the context of petroleum engineering involve gas-liquid flow in a wellbore where there is some interaction with the surrounding reservoir. For an example of such a model in the context of *single-phase* flow we refer to [7, 8] and references therein. In this paper we consider a *two-phase* gas-liquid model with inclusion of well-reservoir interaction. For instance, gas-kick refers to a situation where gas flows into the well from the formation at some regions along the wellbore. As this gas ascends in the well it will typically experience a lower pressure. This leads to decompression of the gas, which in turn, potentially can provoke blow-out like scenarios. In particular, equipment can be placed along the wellbore that allow for some kind of control on the flow between well and formation. In this work we focus on a gas-liquid model where gas is allowed to flow between well and formation governed by a given flow rate function $A(x,t)$.

The dynamics of the two-phase well flow is supposed to be dictated by a compressible gas-liquid model of the drift-flux type. More precisely, it takes the following form

$$\begin{aligned} \partial_t[\alpha_g\rho_g] + \partial_x[\alpha_g\rho_g u_g] &= [\alpha_g\rho_g]A(x,t) \\ \partial_t[\alpha_l\rho_l] + \partial_x[\alpha_l\rho_l u_l] &= 0 \\ \partial_t[\alpha_l\rho_l u_l + \alpha_g\rho_g u_g] + \partial_x[\alpha_g\rho_g u_g^2 + \alpha_l\rho_l u_l^2 + P] &= -q + \partial_x[\varepsilon\partial_x u_{mix}], \quad u_{mix} = \alpha_g u_g + \alpha_l u_l, \end{aligned} \tag{1}$$

where $\varepsilon \geq 0$. This formulation allows us to study transient flows in a well together with a possible flow of gas between well and surrounding reservoir represented by the rate term $A(x,t)$. The model is supposed under isothermal conditions. The unknowns are ρ_l, ρ_g the liquid and gas densities, α_l, α_g volume fractions of liquid and gas satisfying $\alpha_g + \alpha_l = 1$, u_l, u_g velocities of

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liquid and gas, P common pressure for liquid and gas, and q representing external forces like gravity and friction. Since the momentum is given only for the mixture, we need an additional closure law, a so-called hydrodynamical closure law, which connects the two phase fluid velocities. More generally, this law should be able to take into account the different flow regimes. For more general information concerning two-phase flow dynamics we refer to [5, 4, 16], whereas we refer to [6, 13] and references therein for more information concerning numerical methods and some basic mathematical properties of the model (1).

In this work we consider the special case where a no-slip condition is assumed, i.e.,

$$u_g = u_l = u. \tag{2}$$

In previous works [10, 11, 24, 25] a simplified version of the mixture momentum equation of (1) has been used given by

$$\partial_t[\alpha_l\rho_l u_l] + \partial_x[\alpha_l\rho_l u_l^2 + p] = -q + \partial_x[\varepsilon\partial_x u_{mix}], \quad u_{mix} = \alpha_g u_g + \alpha_l u_l, \tag{3}$$

where certain gas related terms have been ignored. In the present work we deal with the full momentum equation of (1), however, still under the assumption of equal fluid velocity (2). Assuming a polytropic gas law relation $p = C\rho_g^\gamma$ with $\gamma > 1$ and incompressible liquid $\rho_l = \text{Const}$ we get a pressure law of the form

$$P(n,m) = C\Big(\frac{n}{\rho_l - m}\Big)^\gamma, \tag{4}$$

where we use the notation $n = \alpha_g\rho_g$ and $m = \alpha_l\rho_l$. In particular, we see that pressure becomes singular at transition to pure liquid phase, i.e., $\alpha_l = 1$ and $\alpha_g = 0$, which yields $m = \rho_l$ and $n = 0$. Another possibility is that the gas density ρ_g vanishes which implies vacuum, i.e., $p = 0$. In order to treat this difficulty we shall consider (1) in a free boundary problem setting where the masses m and n initially occupy only a finite interval $[a,b] \subset \mathbb{R}$. That is,

$$n(x,0) = n_0(x) > 0, \qquad m(x,0) = m_0(x) > 0, \qquad u(x,0) = u_0(x), \qquad x \in [a,b],$$

and $n_0 = m_0 = 0$ outside $[a,b]$. The viscosity coefficient ε is assumed to be a functional of the masses m and n, i.e. $\varepsilon = \varepsilon(n,m)$. More precisely, we assume that

$$\varepsilon(n,m) = D\frac{(n+m)^\beta}{(\rho_l - m)^{\beta+1}}, \qquad \beta \in (0, 1/3), \tag{5}$$

for a constant D, which is a natural generalization of the viscosity coefficient that was used in [10, 24] to the case where we consider the full momentum equation. We refer to [12] for more information concerning the choice of the viscosity coefficient.

Introducing the total mass $\rho = n + m$ and rewriting the model (1) in terms of Lagrangian variables, the free boundaries are converted into fixed and we get a model of the form

$$\begin{aligned} \partial_t n + (n\rho)\partial_x u &= nA \\ \partial_t \rho + \rho^2\partial_x u &= nA \\ \partial_t u + \partial_x P(n,\rho) &= -u\frac{n}{\rho}A + \partial_x(\varepsilon(n,\rho)\rho\partial_x u), \qquad x \in (0,1), \end{aligned} \tag{6}$$

with pressure law

$$P(n,\rho) = \Big(\frac{n}{\rho_l - [\rho - n]}\Big)^\gamma, \tag{7}$$

and viscosity coefficient

$$\varepsilon(n,\rho) = \frac{\rho^\beta}{(\rho_l - [\rho - n])^{\beta+1}}, \qquad \beta \in (0, 1/3), \tag{8}$$

where we have set the constant C, D to be one for simplicity, whereas boundary conditions are

$$P(n,\rho) = \varepsilon(n,\rho)\rho u_x, \qquad \text{at } x = 0, 1, \quad t \geq 0, \tag{9}$$

and initial conditions are

$$n(x,0) = n_0(x), \quad \rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x), \qquad x \in [0,1]. \tag{10}$$

The main novelty of this work compared to the previous recent works [10, 11, 24, 25] on the model (1) is as follows:

- We include the full momentum equation of (1) in contrast to the simplified one given by (2);
- We include well-reservoir interaction in the sense that gas can flow between the well and reservoir. As a consequence, new terms appear in the continuity and momentum equations, see (6).

We obtain an existence result (Theorem 2.1) for the model (6)–(10) for a class of weak solutions under suitable regularity conditions on the initial data n_0, m_0, and u_0 and the well-formation rate function $A(x,t)$. The key point leading to this result is the possibility to obtain sufficient pointwise control on the gas mass n and liquid mass m, upper as well as lower limits. More precisely, by assuming initially that the gas and liquid mass n and m do not disappear or blow up on $[0,1]$, that is,

$$C^{-1} \leq n(x,0) \leq C, \qquad 0 < \mu \leq m(x,0) \leq \rho_l - \mu < \rho_l,$$

for a suitable constant $C > 0$ and $\mu > 0$, then the same will be true for the masses n and m for all $t \in [0,T]$ for any specified time $T > 0$. This nice feature allows us to obtain various estimates which ensure convergence to a class of weak solutions. By imposing more regularity on the fluid velocity we also derive a uniqueness result (Theorem 5.1) in a corresponding smaller class of weak solutions. A main tool in this analysis is the introduction of a suitable variable transformation allowing for application of ideas and techniques similar to those used in [21, 18, 19, 23, 22, 17] in previous studies of the single-phase Navier-Stokes equations. In this sense the approach of this work follows along the same line as [10, 11, 24]. However, in the current work the variable transformation must also account for the fact that the full momentum equation is used as well as ensure that the new terms representing well-formation interaction can be properly handled.

We end this section by a brief review of more recent works on models similar to (1). In [12] we explore existence of global weak solutions for a version of (1) where a physical relevant friction term has been added together with a general pressure law. Furthermore, the model has also been studied in Eulerian coordinates with a simplified momentum equation similar to (3) and constant viscosity coefficient [9]. Existence of global weak solutions was obtained under suitable assumptions on initial data. For a similar result where the model is studied in a 2D setting we refer to [26]. The drift-flux model has also been studied in the context of flow in networks [3]. Finally, we also would like to mention some works on a related multicomponent gas model without viscosity term where discrete algorithms are used to rigorously demonstrate convergence towards a weak solution [14, 15]. A similar type of model with focus on phase transition is studied in [1, 2]. In particular, global existence of weak solutions is shown as well as convergence towards a reduced model.

The rest of this paper is organized as follows. In Section 2 we derive the Lagrangian form of the model (1) and state precisely the main theorem and its assumptions. In Section 3 we describe a priori estimates for an auxiliary model obtained from (6) by using an appropriate variable transformation. In Section 4 we consider approximate solutions to (6) obtained by regularizing initial data. By means of the estimates of Section 3, we get a number of estimates for the approximate solutions of Section 4 which imply compactness. Convergence to a weak solution then follows by standard arguments. Finally, in Section 5 we present a uniqueness result for an appropriate (smaller) class of weak solutions.

2. A global existence result

We focus on the case where the liquid is assumed to be incompressible which implies that we use the pressure law (4). We refer to the works [10, 11] for more details. Moreover, we neglect external force terms (friction and gravity). We then rewrite the model slightly by adding the two continuity equations and introducing the total mass ρ given by

$$\rho = n + m. \tag{11}$$

Hence, we consider the compressible gas-incompressible liquid two-phase model written in the following form:

$$
\begin{aligned}
\partial_t n + \partial_x[nu] &= nA \\
\partial_t \rho + \partial_x[\rho u] &= nA \\
\partial_t[\rho u] + \partial_x[\rho u^2] + \partial_x P(n,\rho) &= \partial_x[\varepsilon(n,\rho)\partial_x u].
\end{aligned}
\tag{12}
$$

with $A = A(x,t)$. Note that this system also takes the form

$$
\begin{aligned}
\partial_t n + \partial_x[nu] &= nA \\
\partial_t \rho + \partial_x[\rho u] &= nA, \\
u(\partial_t \rho + \partial_x[\rho u]) + \rho(\partial_t u + u\partial_x u) + \partial_x P(n,\rho) &= \partial_x[\varepsilon(n,\rho)\partial_x u],
\end{aligned}
\tag{13}
$$

which corresponds to

$$
\begin{aligned}
(\partial_t n + u\partial_x n) + n\partial_x u &= nA \\
(\partial_t \rho + u\partial_x \rho) + \rho\partial_x u &= nA, \\
\rho(\partial_t u + u\partial_x u) + \partial_x P(n,\rho) &= -unA + \partial_x[\varepsilon(n,\rho)\partial_x u].
\end{aligned}
\tag{14}
$$

Here

$$
P(n,\rho) = \Big(\frac{n}{\rho_l - m}\Big)^{\gamma} = \Big(\frac{n}{\rho_l - [\rho - n]}\Big)^{\gamma}, \qquad \gamma > 1, \tag{15}
$$

$$
\varepsilon(n,\rho) = \frac{(n+m)^{\beta}}{(\rho_l - m)^{\beta+1}} = \frac{\rho^{\beta}}{(\rho_l - [\rho - n])^{\beta+1}}, \qquad \beta \in (0, 1/3). \tag{16}
$$

2.1. **Main idea.** The idea of this paper is to study the model (12)–(16) in a setting where sufficient pointwise control on the masses ρ and n can be ensured. Motivated by previous studies of the single-phase Navier-Stokes model [21, 18, 19, 23, 22, 17], we propose to study (12) in a free-boundary setting where the total mass ρ and gas mass n are of compact support initially and connect to the vacuum regions (where $n = \rho = 0$) discontinuously. More precisely, we shall study the Cauchy problem (12) with initial data

$$
(n,\rho,\rho u)(x,0) = \begin{cases} (n_0, \rho_0, \rho_0 u_0) & x \in [a,b], \\ (0,0,0) & \text{otherwise}, \end{cases}
$$

where $\min_{x\in[a,b]} n_0 > 0$, $\min_{x\in[a,b]} \rho_0 > 0$, and $n_0(x), \rho_0(x)$ are in H^1. In other words, we study the two-phase model in a setting where an initial true two-phase mixture region (a,b) is surrounded by vacuums states $n = \rho = 0$ on both sides. Letting $a(t)$ and $b(t)$ denote the particle paths initiating from $(a,0)$ and $(b,0)$, respectively, in the x-t coordinate system, these paths represent free boundaries, i.e., the interface of the gas-liquid mixture and the vacuum. These are determined by the equations

$$
\begin{aligned}
&\frac{d}{dt}a(t) = u(a(t),t), && \frac{d}{dt}b(t) = u(b(t),t), \\
&(-P(n,\rho) + \varepsilon(n,\rho)u_x)\,(a(t)^+, t) = 0, && (-P(n,\rho) + \varepsilon(n,\rho)u_x)\,(b(t)^-, t) = 0.
\end{aligned}
\tag{17}
$$

We introduce a new set of variables (ξ,τ) by using the coordinate transformation

$$
\xi = \int_{a(t)}^{x} \rho(y,t)\,dy, \qquad \tau = t. \tag{18}
$$

Thus, ξ represents a convenient rescaling of x. In particular, the free boundaries $x = a(t)$ and $x = b(t)$, in terms of the new variables ξ and τ, take the form

$$
\tilde{a}(\tau) = 0, \qquad \tilde{b}(\tau) = \int_{a(t)}^{b(t)} \rho(y,t)\,dy = \text{const} \qquad \text{(by assumption)}, \tag{19}
$$

where $\int_a^b \rho_0(y)\,dy$ is the total liquid mass initially, which we normalize to 1. In other words, the interval $[a,b]$ in the x-t system appears as the interval $[0,1]$ in the ξ-τ system.

Remark 2.1. *Note that we implicitly in* (19) *use the assumption*

$$\int_{a(t)}^{b(t)} [nA](y,t)\,dy = 0.$$

This puts a constrain on the well-formation interaction. In particular, it implies that if there is inflow of gas in one region along the well ($A > 0$)*, then there must be outflow in another region* ($A > 0$) *such that the total mass* ρ *is conserved.*

Next, we rewrite the model itself (12) in the new variables (ξ, τ). First, in view of the particle paths $X_\tau(x)$ given by

$$\frac{dX_\tau(x)}{d\tau} = u(X_\tau(x), \tau), \qquad X_0(x) = x,$$

the system (14) now takes the form

$$\begin{aligned} \frac{dn}{d\tau} + nu_x &= nA(x,\tau) \\ \frac{d\rho}{d\tau} + \rho u_x &= nA(x,\tau) \\ \rho\frac{du}{d\tau} + P(n,\rho)_x &= -unA(x,\tau) + (\varepsilon(n,\rho)u_x)_x. \end{aligned}$$

Applying (18) to shift from (x,t) to (ξ,τ) we get

$$\begin{aligned} n_\tau + (n\rho)u_\xi &= nA(x(\xi,\tau),\tau) \\ \rho_\tau + (\rho^2)u_\xi &= nA(x(\xi,\tau),\tau) \\ u_\tau + P(n,\rho)_\xi &= -u\frac{n}{\rho}A(x(\xi,\tau),\tau) + (\varepsilon(n,\rho)\rho u_\xi)_\xi, \qquad \xi \in (0,1), \quad \tau \geq 0, \end{aligned}$$

where $x(\xi,\tau) = a(\tau) + \int_0^\xi \rho^{-1}(y,\tau)\,dy$ for $\xi \in [0,1]$ and with boundary conditions, in view of (17), given by

$$P(n,\rho) = \varepsilon(n,\rho)\rho u_\xi, \qquad \text{at } \xi = 0,1, \quad \tau \geq 0.$$

In addition, we have the initial data

$$n(\xi,0) = n_0(\xi), \quad \rho(\xi,0) = \rho_0(\xi), \quad u(\xi,0) = u_0(\xi), \qquad \xi \in [0,1].$$

In the following we replace the coordinates (ξ,τ) by (x,t) such that the model now takes the form

$$\begin{aligned} \partial_t n + (n\rho)\partial_x u &= nA(x,t) \\ \partial_t \rho + \rho^2 \partial_x u &= nA(x,t) \\ \partial_t u + \partial_x P(n,\rho) &= -u\frac{n}{\rho}A(x,t) + \partial_x(E(n,\rho)\partial_x u), \qquad x \in (0,1), \end{aligned} \tag{20}$$

with

$$P(n,\rho) = \Big(\frac{n}{\rho_l - [\rho - n]}\Big)^\gamma, \qquad \gamma > 1, \tag{21}$$

and

$$E(n,\rho) = \Big(\frac{\rho}{\rho_l - [\rho - n]}\Big)^{\beta+1}, \qquad 0 < \beta < 1/3. \tag{22}$$

Moreover, boundary conditions are given by

$$P(n,\rho) = E(n,\rho)u_x, \qquad \text{at } x = 0,1, \quad t \geq 0, \tag{23}$$

whereas initial data are

$$n(x,0) = n_0(x), \quad \rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x), \qquad x \in [0,1]. \tag{24}$$

We observe that the model problem (20)–(24) coincides with the model (6)–(10) stated in the introduction part. We shall in the following assume that the external controlled flow rate function $A(x,t)$ satisfies some estimates, essentially, that it is bounded and its spatial derivative is in L^2. This is precisely stated below.

2.2. **Main result.** Before we state the main result for the model (20)–(24), we describe the notation we apply throughout the paper. $W^{1,2}(I) = H^1(I)$ represents the usual Sobolev space defined over $I = (0,1)$ with norm $\|\cdot\|_{W^{1,2}}$. Moreover, $L^p(K,B)$ with norm $\|\cdot\|_{L^p(K,B)}$ denotes the space of all strongly measurable, pth-power integrable functions from K to B where K typically is subset of $\mathbb{R}$ and B is a Banach space. In addition, let $C^\alpha[0,1]$ for $\alpha \in (0,1)$ denote the Banach space of functions on $[0,1]$ which are uniformly Hölder continuous with exponent α. Similarly, let $C^{\alpha,\alpha/2}(D_T)$ represent the Banach space of functions on $D_T = [0,1]\times[0,T]$ which are uniformly Hölder continuous with exponent α in x and $\alpha/2$ in t.

Theorem 2.1 (Main Result)**.** *Assume that $\gamma > 1$ and $\beta \in (0,1/3)$ respectively in (21) and (22), and that the initial data (n_0, m_0, u_0) satisfy*

(i) $\inf_{[0,1]} n_0 > 0$, $\sup_{[0,1]} n_0 < \infty$, $\inf_{[0,1]} m_0 > 0$, *and* $\sup_{[0,1]} m_0 < \rho_l$;
(ii) $n_0, m_0 \in W^{1,2}(I)$;
(iii) $u_0 \in L^{2q}(I)$, *for* $q \in \mathbb{N}$.

As a consequence, the function $c_0 = \frac{n_0}{n_0+m_0}$ *satisfies that*

$$\inf_{[0,1]} c_0 > 0, \qquad \sup_{[0,1]} c_0 < 1, \qquad c_0 \in W^{1,2}(I). \tag{25}$$

Moreover, the function $Q_0 = \frac{n_0+m_0}{\rho_l - m_0}$ *satisfies that*

$$\inf_{[0,1]} Q_0 > 0, \qquad \sup_{[0,1]} Q_0 < \infty, \qquad Q_0 \in W^{1,2}(I). \tag{26}$$

In addition, the well-formation flow rate function $A(x,t)$ is assumed to satisfy for all times $t \geq 0$

(iv) $\sup_{x\in[0,1]} |A(x,t)| \leq M < \infty$;
(v) $A(\cdot,t) \in W^{1,2}(I)$;
(vi) $A(0,t) = 0$.

Then the initial-boundary problem (20)–(24) possesses a global weak solution (n,ρ,u) in the sense that for any $T > 0$, the following holds:

(A) *We have the following estimates:*

$$n,\rho \in L^\infty([0,T],W^{1,2}(I)), \qquad n_t,\rho_t \in L^2([0,T],L^2(I)),$$
$$u \in L^\infty([0,T],L^{2q}(I)) \cap L^2([0,T],H^1(I)).$$

More precisely, we have $\forall (x,t) \in [0,1]\times[0,T]$ that

$$\begin{aligned}
&0 < \inf_{x\in[0,1]} c(x,t), \qquad \sup_{x\in[0,1]} c(x,t) < 1, \qquad c := \frac{n}{\rho},\\
&0 < \mu \inf_{x\in[0,1]}(c) \leq n(x,t) \leq \Big(\frac{\rho_l - \mu}{1 - \sup_{x\in[0,1]}(c)}\Big) \sup_{x\in[0,1]}(c),\\
&0 < \mu \leq \rho \leq \frac{\rho_l - \mu}{1 - \sup_{x\in[0,1]}(c)},
\end{aligned} \tag{27}$$

for a non-negative constant $\mu = \mu(\|c_0\|_{W^{1,2}(I)}, \|Q_0^\beta\|_{W^{1,2}(I)}, \|A\|_{W^{1,2}(I)}, \|u_0\|_{L^{2k}(I)}, \inf_{[0,1]} c_0, \sup_{[0,1]} c_0, \inf_{[0,1]} Q_0, \sup_{[0,1]} Q_0, M, T) > 0$.

(B) *Moreover, the following equations hold,*

$$\begin{aligned}
&n_t + n\rho u_x = nA, \quad \rho_t + \rho^2 u_x = nA,\\
&(n,\rho)(x,0) = (n_0(x),\rho_0(x)), \ \text{for a.e. } x \in (0,1) \text{ and any } t \geq 0,\\
&\int_0^\infty \int_0^1 \Big[u\phi_t + \Big(P(n,\rho) - E(n,\rho)u_x\Big)\phi_x - u\frac{n}{\rho}A\phi\Big]\,dx\,dt + \int_0^1 u_0(x)\phi(x,0)\,dx = 0
\end{aligned} \tag{28}$$

for any test function $\phi(x,t) \in C_0^\infty(D)$, with $D := \{(x,t)\,|\, 0 \leq x \leq 1,\ t \geq 0\}$.

The proof of Theorem 2.1 is based on a priori estimates for the approximate solutions of (20)–(24) and a corresponding limit procedure. In particular, it is possible to obtain pointwise upper and lower limits for ρ that allows us to control the quantities $\int_0^1 (\rho_x)^2\,dx$ and $\int_0^1 (n_x)^2\,dx$, see Corollary 3.3. A main idea in the analysis is to employ the quantity $Q(n,\rho) = \rho/(\rho_l - [\rho - n])$, which connects pressure $P(n,\rho)$ and viscosity coefficient $E(n,\rho)$, and reformulate the model (20) in terms of the variables (c, Q, u) where $c = n/\rho$. Together with higher order regularity of u and $(Q^\beta)_x$, and energy-conservation obtained by adopting techniques used in [21, 18, 19, 23, 22, 17] for single-phase Navier-Stokes equations, pointwise upper and lower limits for $Q(n,\rho)$ can be derived. This, in turn, gives the required boundedness on ρ from below and above together with the L^2 estimate of n_x and ρ_x. Armed with these estimates we can rely on standard compactness arguments to prove Theorem 2.1. This is done in Section 4.

Special challenges we have to deal with in this work, compared to the previous two-phase works [10, 11, 24, 25] where a similar approach was employed, are:

- The variable $c = c(x,t)$ becomes time-dependent as a consequence of the well-formation interaction. This makes some of the estimates more involved, e.g. manifested by the appearance of Lemma 3.2, which does not appear in [10, 11, 24, 25].
- The result of Lemma 3.3 requires a certain regularity on the flow rate function $A(x,t)$.
- The proof of Lemma 3.4 must be extended by new arguments (compared to e.g. [10]) in order to treat new terms representing the well-formation effects.

3. Estimates

Below we derive a priori estimates for (n,ρ,u) which are assumed to be a smooth solution of (20)–(24). We then construct the approximate solutions of (20) in Section 4 by mollifying the initial data n_0, ρ_0, u_0 and obtain global existence by taking the limit.

More precisely, similar to [17, 10] we first assume that (n,ρ,u) is a solution of (20)–(24) on $[0,T]$ satisfying

$$\begin{aligned} &n, n_t, n_x, n_{tx}, \rho, \rho_x, \rho_t, \rho_{tx}, u, u_x, u_t, u_{xx} \in C^{\alpha,\alpha/2}(D_T) \qquad \text{for some } \alpha \in (0,1), \\ &n(x,t) > 0, \qquad \rho(x,t) > 0, \qquad [\rho - n](x,t) < \rho_l \quad \text{on } D_T = [0,1]\times[0,T]. \end{aligned} \tag{29}$$

In the following we will frequently take advantage of the fact that the model (20) can be rewritten in a form more amenable for deriving various useful estimates. We first describe this reformulation, and then present a number of a priori estimates.

3.1. **A reformulation of the model** (20).

We introduce the variable

$$c = \frac{n}{\rho}, \tag{30}$$

and see that (20) corresponds to

$$\begin{aligned} \rho\partial_t c + c\partial_t\rho + [c\rho^2]\partial_x u &= [c\rho]A \\ \partial_t\rho + \rho^2\partial_x u &= [c\rho]A \\ \partial_t u + \partial_x P(c,\rho) &= -ucA + \partial_x(E(c,\rho)\partial_x u), \end{aligned}$$

that is,

$$\begin{aligned} \rho\partial_t c + c[c\rho]A &= [c\rho]A \\ \partial_t\rho + \rho^2\partial_x u &= [c\rho]A \\ \partial_t u + \partial_x P(c,\rho) &= -ucA + \partial_x(E(c,\rho)\partial_x u), \end{aligned}$$

which, in turn can be reformulated as

$$\begin{aligned} \partial_t c = c(1-c)A = ckA, \qquad k = k(x,t) &:= 1 - c(x,t), \\ \partial_t\rho + \rho^2\partial_x u &= c\rho A \\ \partial_t u + \partial_x P(c,\rho) &= -ucA + \partial_x(E(c,\rho)\partial_x u), \end{aligned} \tag{31}$$

with

$$P(c,\rho) = c^{\gamma}\Big(\frac{\rho}{\rho_l - k(x,t)\rho}\Big)^{\gamma}, \qquad k(x,t) = 1 - c(x,t) \qquad \gamma > 1, \tag{32}$$

and

$$E(c,\rho) = \Big(\frac{\rho}{\rho_l - k(x,t)\rho}\Big)^{\beta+1}, \qquad 0 < \beta < 1/3. \tag{33}$$

Moreover, boundary conditions are given by

$$P(c,\rho) = E(c,\rho)u_x, \qquad \text{at } x = 0,1, \quad t \geq 0, \tag{34}$$

whereas initial data are

$$c(x,0) = c_0(x), \quad \rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x), \qquad x \in [0,1]. \tag{35}$$

Corollary 3.1. *Under the assumptions of Theorem 2.1, it follows that for a given time $T > 0$*

$$0 < \inf_{x\in[0,1]} c(x,t), \qquad \sup_{x\in[0,1]} c(x,t) < 1. \tag{36}$$

Proof. Note that from (31) we have

$$c_t = c(1-c)A(x,t),$$

which corresponds to

$$\frac{1}{c(1-c)}c_t = A(x,t), \qquad c \in (0,1),$$

i.e.

$$G(c)_t = A(x,t), \qquad G(c) = \log\Big(\frac{c}{1-c}\Big).$$

This implies that

$$\frac{c(x,t)}{1-c(x,t)} = \frac{c_0(x)}{1-c_0(x)}\exp\Big(\int_0^t A(x,s)\,ds\Big).$$

Note also that the inverse of $h(c) = c/(1-c)$ is $h^{-1}(d) = d/(1+d)$, such that $h^{-1} : [0,\infty) \to [0,1)$ and is one-to-one. Consequently,

$$c(x,t) = h^{-1}\Big(\frac{c_0(x)}{1-c_0(x)}\exp\Big(\int_0^t A(x,s)\,ds\Big)\Big), \tag{37}$$

and $0 < c(x,t) < 1$ for $c_0(x) \in (0,1)$. In particular, we see that if

$$0 < \inf_{[0,1]} c_0(x), \qquad \sup_{[0,1]} c_0(x) < 1, \qquad \sup_{[0,1]} |A(x,t)| \leq M, \tag{38}$$

which follows from the assumptions on n_0, m_0, and A given in Theorem 2.1, the conclusion (36) holds. □

In order to obtain the a priori estimates it will be convenient to introduce a new reformulation of the model (31)–(35). This reformulation allows us to deal with the potential singular behavior associated with the pressure law (32) and viscosity coefficient (33). A similar approach was used in [10, 11, 24]. However, compared to those works we now also have to take into account additional terms due to the well-formation interaction and the fact that a full momentum equation is used in the model. For that purpose, we introduce the variable

$$Q(\rho,k) = \frac{\rho}{\rho_l - k(x,t)\rho}, \tag{39}$$

and observe that

$$\rho = \frac{\rho_l Q}{1+kQ}, \qquad \frac{1}{\rho} = \frac{1}{\rho_l Q} + \frac{k}{\rho_l}. \tag{40}$$

Consequently, we get

$$\begin{aligned}
Q(\rho,k)_t &= Q_\rho \rho_t + Q_k k_t \\
&= \Big(\frac{1}{\rho_l - k\rho} + \frac{\rho k}{(\rho_l - k\rho)^2}\Big)\rho_t + \frac{\rho^2}{(\rho_l - k\rho)^2} k_t \\
&= \frac{\rho_l}{(\rho_l - k\rho)^2}\rho_t + \frac{\rho^2}{(\rho_l - k\rho)^2} k_t \\
&= \frac{\rho_l}{(\rho_l - k\rho)^2}[c\rho A - \rho^2 u_x] + \frac{\rho^2}{(\rho_l - k\rho)^2} k_t \qquad \text{(using second equation of (31))} \\
&= \frac{\rho_l c\rho A}{(\rho_l - k\rho)^2} - \frac{\rho_l \rho^2}{(\rho_l - k\rho)^2} u_x + Q^2 k_t \\
&= \frac{\rho_l c\rho^2 A}{\rho(\rho_l - k\rho)^2} - \rho_l Q^2 u_x - Q^2 c_t \\
&= \rho_l cA\Big(\frac{1}{\rho_l Q} + \frac{k}{\rho_l}\Big)Q^2 - \rho_l Q^2 u_x - Q^2 ckA \qquad \text{(using (40) and first equation of (31))} \\
&= cA\Big(Q + kQ^2\Big) - \rho_l Q^2 u_x - Q^2 ckA \\
&= cAQ + cAkQ^2 - \rho_l Q^2 u_x - Q^2 ckA \\
&= cAQ - \rho_l Q^2 u_x.
\end{aligned}$$

Thus, we may rewrite the model (31) in the following form

$$\begin{aligned}
\partial_t c &= kcA \\
\partial_t Q + \rho_l Q^2 u_x &= cAQ \\
\partial_t u + \partial_x P(c,Q) &= -ucA + \partial_x (E(Q)\partial_x u),
\end{aligned} \tag{41}$$

with

$$P(c,Q) = c^\gamma Q(\rho,k)^\gamma, \qquad \gamma > 1, \tag{42}$$

and

$$E(Q) = Q(\rho,k)^{\beta+1}, \qquad 0 < \beta < 1/3. \tag{43}$$

This model is then subject to the boundary conditions

$$P(c,Q) = E(Q)u_x, \qquad \text{at } x = 0,1, \quad t \ge 0. \tag{44}$$

In addition, we have the initial data

$$c(x,0) = c_0(x), \quad Q(x,0) = Q_0(x), \quad u(x,0) = u_0(x), \qquad x = [0,1]. \tag{45}$$

Note that there is a fine tuned balance which leads to the transformed model (41). In particular, the cancelation of the term ckQ^2A appearing in the equation for Q and shown in the above calculation, seems to be crucial for the energy estimate. Note also the new term in the momentum equation accounting for the change in fluid velocity due to inflow/outflow.

3.2. **A priori estimates.** Now we derive a priori estimates for (c,Q,u) by making use of the reformulated model (41)–(45).

Lemma 3.1 (Energy estimate)**.** *We have the basic energy estimate*

$$\int_0^1 \Big(\frac{1}{2}u^2 + \frac{c^\gamma Q(\rho,k)^{\gamma-1}}{\rho_l(\gamma-1)}\Big)(x,t)\,dx + \int_0^t\int_0^1 Q(\rho,k)^{\beta+1}(u_x)^2\,dx\,ds \le C_1 \tag{46}$$

where $C_1 = C_1(\sup_{[0,1]} Q_0, \|u_0\|_{L^2(I)}, \|c_0\|_{L^\gamma(I)}, M)$*. Moreover,*

$$Q(\rho,k)(x,t) \le C_2, \qquad \forall (x,t) \in [0,1]\times[0,T], \tag{47}$$

where $C_2 = C_2(\sup_{[0,1]} Q_0, \|u_0\|_{L^2(I)}, \|c_0\|_{L^\gamma(I)}, M, T)$*. Moreover, for any positive integer* q*,*

$$\int_0^1 u^{2q}(x,t)\,dx + q(2q-1)\int_0^t\int_0^1 u^{2q-2}Q(\rho,k)^{1+\beta}(u_x)^2\,dx\,dt \le C_3, \tag{48}$$

where $C_3 = C_3(\|u_0\|_{L^{2q}(I)}, T, q, C_2, M)$.

Proof. We consider the proof in three steps.

Estimate (46)**:** We multiply the third equation of (41) by u and integrate over $[0,1]$ in space. We apply the boundary condition (44) and the equation

$$\frac{c^\gamma}{\rho_l(\gamma-1)}(Q^{\gamma-1})_t + c^\gamma Q^\gamma u_x = \frac{1}{\rho_l}c^{\gamma+1}Q^{\gamma-1}A, \tag{49}$$

obtained from the second equation of (41) by multiplying with $c^\gamma Q^{\gamma-2}$. This equation also corresponds to

$$\frac{1}{\rho_l(\gamma-1)}(c^\gamma Q^{\gamma-1})_t - \frac{Q^{\gamma-1}}{\rho_l(\gamma-1)}(c^\gamma)_t + c^\gamma Q^\gamma u_x = \frac{1}{\rho_l}c^{\gamma+1}Q^{\gamma-1}A, \tag{50}$$

which in turn can be rewritten as

$$\frac{1}{\rho_l(\gamma-1)}(c^\gamma Q^{\gamma-1})_t - \frac{\gamma}{\rho_l(\gamma-1)}Q^{\gamma-1}c^\gamma kA + P(c,Q)u_x = \frac{1}{\rho_l}c^{\gamma+1}Q^{\gamma-1}A, \tag{51}$$

where we have used the first equation of (41). Then, we get

$$\begin{aligned}\frac{d}{dt}\int_0^1\Big(\frac12 u^2 + \frac{c^\gamma Q^{\gamma-1}}{\rho_l(\gamma-1)}\Big)\,dx - \int_0^1 \frac{\gamma c^\gamma Q^{\gamma-1}}{\rho_l(\gamma-1)}[kA]\,dx + \int_0^1 u^2[cA]\,dx\\ + \int_0^1 E(Q)(u_x)^2\,dx = \frac{1}{\rho_l}\int_0^1 c^{\gamma+1}Q^{\gamma-1}A\,dx = \frac{1}{\rho_l}\int_0^1 c^\gamma Q^{\gamma-1}[cA]\,dx.\end{aligned}$$

Using that $|kA(x,t)|, |cA(x,t)| \le M$, in view of the assumptions of Theorem 2.1 and the result of Corollary 3.1, application of Gronwall's inequality, respectively, for the term $\int_0^1 u^2[cA]\,dx$, $\int_0^1 c^\gamma Q^{\gamma-1}[cA]\,dx$, and $\int_0^1 c^\gamma Q^{\gamma-1}[kA]\,dx$, gives (46).

Estimate (47)**:** From the second equation of (41) we deduce the equation

$$\frac{1}{\rho_l}(Q^\beta)_t + \beta Q^{\beta+1}u_x = \frac{\beta}{\rho_l}cQ^\beta A. \tag{52}$$

Integrating over $[0,t]$, we get

$$Q^\beta(x,t) = Q^\beta(x,0) - \beta\rho_l\int_0^t Q^{\beta+1}u_x\,ds + \beta\int_0^t cQ^\beta A\,ds. \tag{53}$$

Then, we integrate the third equation of (41) over $[0,x]$ and get

$$\int_0^x u_t(y,t)\,dy + P(c,Q) - P(c(0,t),Q(0,t)) + (E(Q)u_x)(0,t) + \int_0^x ucA\,dy = E(Q)u_x = Q^{\beta+1}u_x.$$

Using the boundary condition (44) and inserting the above relation into the right hand side of (53), we get

$$\begin{aligned}&Q^\beta(x,t)\\ &= Q^\beta(x,0) - \beta\rho_l\int_0^t\Big(\int_0^x u_t(y,t)\,dy + P(c,Q) + \int_0^x ucA\,dy\Big)\,ds + \beta\int_0^t cQ^\beta A\,ds\\ &= Q^\beta(x,0) - \beta\rho_l\int_0^x (u(y,t)-u_0(y))\,dy - \beta\rho_l\int_0^t P(c,Q)\,ds\\ &\qquad - \beta\rho_l\int_0^t\int_0^x u[cA]\,dy\,ds + \beta\int_0^t Q^\beta[cA]\,ds.\end{aligned} \tag{54}$$

Consequently, since $P(c,Q) \geq 0$ and $|cA| \leq M$

$$Q^\beta(x,t) \leq Q^\beta(x,0) + \beta\rho_l \int_0^1 |u(y,t)|\,dy + \beta\rho_l \int_0^1 |u_0(y)|\,dy \\ + \beta\rho_l M \int_0^t \int_0^x |u|\,dy\,ds + \beta M \int_0^t Q^\beta(x,s)\,ds.$$

Applying Hölder's inequality and (46) we can bound $\int_0^1 |u|\,dy$. Moreover, the term $\int_0^t Q^\beta\,ds$ can be handled by means of Gronwall's inequality, and the upper bound (47) then follows.

Estimate (48)**:** Multiplying the third equation of (41) by $2qu^{2q-1}$, integrating over $[0,1]\times[0,t]$ and integration by parts together with application of the boundary conditions (44), we get

$$\begin{aligned} &\int_0^1 u^{2q}\,dx + 2q(2q-1)\int_0^t\int_0^1 Q(\rho,k)^{\beta+1}(u_x)^2 u^{2q-2}\,dx\,ds \\ &\quad= \int_0^1 u_0^{2q}\,dx + 2q(2q-1)\int_0^t\int_0^1 c^\gamma Q(\rho,k)^\gamma u^{2q-2}u_x\,dx\,ds - 2q\int_0^t\int_0^1 [cA]u^{2q}\,dx\,ds. \end{aligned} \tag{55}$$

For the second term on the right hand side of (55) we apply Cauchy's inequality with ε, $ab \leq (1/4\varepsilon)a^2 + \varepsilon b^2$, and get

$$\begin{aligned} &\int_0^t\int_0^1 c^\gamma Q(\rho,k)^\gamma u^{2q-2}u_x\,dx\,ds \\ &\quad\leq \frac{1}{4\varepsilon}\int_0^t\int_0^1 c^{2\gamma}Q(\rho,k)^{2\gamma-\beta-1}u^{2q-2}\,dx\,ds + \varepsilon\int_0^t\int_0^1 Q(\rho,k)^{\beta+1}u^{2q-2}(u_x)^2\,dx\,ds \\ &\quad\leq \frac{1}{4\varepsilon}\sup_{x\in[0,1]}(c^{2\gamma})\int_0^t\int_0^1 Q(\rho,k)^{2\gamma-\beta-1}u^{2q-2}\,dx\,ds + \varepsilon\int_0^t\int_0^1 Q(\rho,k)^{\beta+1}u^{2q-2}(u_x)^2\,dx\,ds. \end{aligned}$$

The last term clearly can be absorbed in the second term of the left-hand side of (55) by the choice $\varepsilon = 1/2$. Finally, let us see how we can bound the term $\int_0^t\int_0^1 u^{2q-2}Q(\rho,k)^{2\gamma-1-\beta}\,dx\,ds$. In view of Young's inequality $ab \leq (1/p)a^p + (1/r)b^r$ where $1/p + 1/r = 1$, we get for the choice $p = q$ and $r = q/(q-1)$

$$\begin{aligned} \int_0^t\int_0^1 u^{2q-2}Q(\rho,k)^{2\gamma-1-\beta}\,dx\,ds &\leq \frac{1}{q}\int_0^t\int_0^1 Q(\rho,k)^{(2\gamma-1-\beta)q}\,dx\,ds + \frac{q-1}{q}\int_0^t\int_0^1 u^{2q}\,dx\,ds \\ &\leq \frac{C_2^{(2\gamma-1-\beta)q}}{q}t + \frac{q-1}{q}\int_0^t\int_0^1 u^{2q}\,dx\,ds, \end{aligned}$$

by using (47). To sum up, we get

$$\begin{aligned} &\int_0^1 u^{2q}\,dx + q(2q-1)\int_0^t\int_0^1 Q(\rho,k)^{\beta+1}(u_x)^2u^{2q-2}\,dx\,ds \\ &\leq \int_0^1 u_0^{2q}\,dx + 2q(2q-1)\frac{1}{4\varepsilon}\sup_{x\in[0,1]}(c^{2\gamma})\Big[\frac{C_2^{2\gamma-1-\beta}}{q}t + \frac{q-1}{q}\int_0^t\int_0^1 u^{2q}\,dx\,ds\Big] + 2qM\int_0^t\int_0^1 u^{2q}\,dx\,ds \\ &= \int_0^1 u_0^{2q}\,dx + (2q-1)\sup_{x\in[0,1]}(c^{2\gamma})\Big[C_2^{2\gamma-1-\beta}t + (q-1)\int_0^t\int_0^1 u^{2q}\,dx\,ds\Big] + 2qM\int_0^t\int_0^1 u^{2q}\,dx\,ds. \end{aligned} \tag{56}$$

In view of Corollary 3.1, application of Gronwall's inequality then allows us to handle the term $\int_0^t\int_0^1 u^{2q}\,dx\,ds$ appearing twice on the right hand side of (56). Hence, the estimate (48) follows. □

The next lemma describes under which conditions $c(x,t)$ is in $W^{1,2}(I)$.

Lemma 3.2 (Additional regularity)**.** *We have the estimate*

$$\int_0^1 (\partial_x c)^2\,dx \leq C_4, \tag{57}$$

for a constant $C_4 = C_4(M, \|c_0\|_{W^{1,2}(I)}, \|A\|_{W^{1,2}(I)}, T)$.

Proof. We set $w = c_x$ and derive from the first equation of (41)

$$w_t = w(1-c)A - cwA + ckA_x = w(1-2c)A + ckA_x.$$

Hence, multiplying by w and integrating over $[0,1]$ we get

$$\int_0^1 (\frac{1}{2}w^2)_t \, dx = \int_0^1 (1-2c)Aw^2 \, dx + \int_0^1 ckA_x w \, dx. \tag{58}$$

Clearly, in view of the assumptions on the flow rate A and the bound on c from Corollary 3.1, we see that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 w^2 \, dx &= \int_0^1 (1-2c)Aw^2 \, dx + \int_0^1 ckA_x w \, dx \\
&\le M\int_0^1 w^2 \, dx + \frac{1}{2}\int_0^1 A_x^2 \, dx + \frac{1}{2}\int_0^1 [ck]^2 w^2 \, dx \\
&\le (M+1)\int_0^1 w^2 \, dx + C,
\end{aligned}$$

where we have used Cauchy's inequality. We conclude, by Gronwall's inequality, that

$$\|c_x\|^2_{L^2(I)} \le C_4,$$

where $C_4 = C_4(M, \|c_0\|_{W^{1,2}(I)}, \|A\|_{W^{1,2}(I)}, T)$. □

The following lemma was also employed in previous works [10, 24]. However, the fact that c is time dependent makes the result more involved, and we need the result of Lemma 3.2.

Lemma 3.3 (Additional regularity)**.** *We have the estimate*

$$\int_0^1 (\partial_x Q^\beta(\rho,k))^2 \, dx \le C_5, \tag{59}$$

for a constant $C_5 = C_5(\|Q_0^\beta\|_{W^{1,2}(I)}, \|c_0\|_{W^{1,2}(I)}, \|u_0\|_{L^2(I)}, C_1, C_2, C_4, M, T)$.

Proof. Using (52) in the third equation of (41) and integrating in time over $[0,t]$ we arrive at

$$\begin{aligned}
u(x,t) - u_0(x) + \int_0^t \partial_x P(c,Q)(x,s) \, ds &= -\int_0^t [cA]u \, ds + \int_0^t \partial_x(E(Q)\partial_x u) \, ds \\
&= -\int_0^t [cA]u \, ds - \frac{1}{\beta\rho_l}(\partial_x Q^\beta(x,t) - \partial_x Q^\beta(x,0)) + \frac{1}{\rho_l}\int_0^t \partial_x([cA]Q^\beta) \, ds.
\end{aligned} \tag{60}$$

Multiplying (60) by $\beta\rho_l(\partial_x Q^\beta)$ and integrating over $[0,1]$ in x, we get

$$\begin{aligned}
\int_0^1 (\partial_x Q^\beta)^2 \, dx &= \int_0^1 (\partial_x Q^\beta)\partial_x Q_0^\beta \, dx - \beta\rho_l \int_0^1 (\partial_x Q^\beta)\Big[(u-u_0) + \int_0^t \partial_x P(c,Q) \, ds\Big] \, dx \\
&\quad + \beta\rho_l \int_0^1 (\partial_x Q^\beta)\Big[-\int_0^t [cA]u \, ds + \frac{1}{\rho_l}\int_0^t \partial_x([cA]Q^\beta) \, ds\Big] \, dx \\
&\le \Big(\int_0^1 (\partial_x Q^\beta)^2 \, dx\Big)^{1/2}\Big(\|\partial_x Q_0^\beta\|_{L^2(I)} + \beta\rho_l\|u-u_0\|_{L^2(I)} + \beta\rho_l\Big\|\int_0^t \partial_x P \, ds\Big\|_{L^2(I)} \\
&\quad + \beta\rho_l\Big\|\int_0^t [cA]u \, ds\Big\|_{L^2(I)} + \beta\Big\|\int_0^t \partial_x([cA]Q^\beta) \, ds\Big\|_{L^2(I)}\Big) \\
&:= ab,
\end{aligned} \tag{61}$$

where we have used Hölder's inequality. Cauchy's inequality $ab \leq a^2/2 + b^2/2$ then gives

$$
\begin{aligned}
&\int_0^1 (\partial_x Q^\beta)^2 \, dx \\
&\quad \leq \frac{1}{2} \int_0^1 (\partial_x Q^\beta)^2 \, dx + \frac{1}{2}\Big(\|\partial_x Q_0^\beta\|_{L^2(I)} + \beta\rho_l \|u - u_0\|_{L^2(I)} + \beta\rho_l \Big\| \int_0^t \partial_x P \, ds \Big\|_{L^2(I)} \\
&\qquad\qquad + \beta\rho_l \Big\| \int_0^t [cA] u \, ds \Big\|_{L^2(I)} + \beta \Big\| \int_0^t \partial_x([cA]Q^\beta) \, ds \Big\|_{L^2(I)} \Big)^2 \qquad (62) \\
&\quad \leq \frac{1}{2} \int_0^1 (\partial_x Q^\beta)^2 \, dx + C + \beta\rho_l T \int_0^t \int_0^1 (\partial_x P)^2 \, dx \, ds \\
&\qquad + \beta T \int_0^t \int_0^1 (\partial_x([cA]Q^\beta))^2 \, dx \, ds + \beta\rho_l T \int_0^t \int_0^1 ([cA]u)^2 \, dx \, ds,
\end{aligned}
$$

by using Hölder's inequality and (48) with $q = 1$ and where $C = C(\|Q_0^\beta\|_{W^{1,2}(I)}, \|u_0\|_{L^2(I)}, C_1)$. Moreover,

$$
\begin{aligned}
\int_0^t \int_0^1 (\partial_x P)^2 \, dx \, ds &= \int_0^t \int_0^1 \Big(Q^\gamma (c^\gamma)_x + c^\gamma (Q^\gamma)_x \Big)^2 \, dx \, ds \\
&\leq 2\Big(\int_0^t \int_0^1 Q^{2\gamma} (c^\gamma)_x^2 \, dx \, ds + \int_0^t \int_0^1 c^{2\gamma} (Q^\gamma)_x^2 \, dx \, ds \Big) \\
&\leq 2(\sup_{x\in[0,1]} Q)^{2\gamma} \int_0^t \int_0^1 (c^\gamma)_x^2 \, dx \, ds + 2(\sup_{x\in[0,1]} c)^{2\gamma} \int_0^t \int_0^1 (Q^\gamma)_x^2 \, dx \, ds \\
&\leq 2C_2^{2\gamma} \int_0^t \int_0^1 (c^\gamma)_x^2 \, dx \, ds + 2 \int_0^t \int_0^1 (Q^\gamma)_x^2 \, dx \, ds,
\end{aligned}
\tag{63}
$$

in view of estimate (47) and Corollary 3.1. Moreover,

$$
\begin{aligned}
\int_0^t \int_0^1 (Q^\gamma)_x^2 \, dx \, ds &= \Big(\frac{\gamma}{\beta}\Big)^2 \int_0^t \int_0^1 Q^{2(\gamma-\beta)} ([Q^\beta]_x)^2 \, dx \, ds \\
&\leq \Big(\frac{\gamma}{\beta}\Big)^2 C_2^{2(\gamma-\beta)} \int_0^t \int_0^1 ([Q^\beta]_x)^2 \, dx \, ds
\end{aligned}
\tag{64}
$$

and

$$
\begin{aligned}
\int_0^t \int_0^1 (c^\gamma)_x^2 \, dx \, ds &= \gamma^2 \int_0^t \int_0^1 c^{2(\gamma-1)} (c_x)^2 \, dx \, ds \\
&\leq \gamma^2 (\sup_{x\in[0,1]} c)^{2(\gamma-1)} \int_0^t \int_0^1 (c_x)^2 \, dx \, ds \leq \gamma^2 t C_4,
\end{aligned}
\tag{65}
$$

in light of Corollary 3.1 and Lemma 3.2. Furthermore, due to the well-reservoir interaction we must also estimate the following term

$$
\begin{aligned}
\int_0^t \int_0^1 (\partial_x([cA]Q^\beta))^2 \, dx \, ds &= \int_0^t \int_0^1 \Big([cA]_x Q^\beta + [cA](Q^\beta)_x \Big)^2 \, dx \, ds \\
&\leq 2 \int_0^t \int_0^1 ([cA]_x)^2 Q^{2\beta} \, dx \, ds + 2 \int_0^t \int_0^1 [cA]^2 ([Q^\beta]_x)^2 \, dx \, ds \\
&\leq 2C_2^{2\beta} \int_0^t \int_0^1 ([cA]_x)^2 \, dx \, ds + 2M^2 \int_0^t \int_0^1 ([Q^\beta]_x)^2 \, dx \, ds \\
&\leq C + 2M^2 \int_0^t \int_0^1 ([Q^\beta]_x)^2 \, dx \, ds,
\end{aligned}
\tag{66}
$$

where we have used that Corollary 3.1, Lemma 3.2, and the assumptions on A imply that $[cA] \in W^{1,2}(I)$. Moreover, for the last term on the right hand side of (62) we have

$$\int_0^t \int_0^1 ([cA]u)^2 \, dx \, ds \leq M^2 \int_0^t \int_0^1 u^2 \, dx \, ds \leq M^2 T C_1. \tag{67}$$

In light of (63)–(67), we conclude from (62) that

$$\int_0^1 (\partial_x Q^\beta)^2 \, dx \leq C + C \int_0^t \int_0^1 (\partial_x Q^\beta)^2 \, dx \, ds.$$

Thus, application of Gronwall's inequality gives the estimate (59). □

The result of the next lemma is crucial. Again we follow along the idea of previous works [17, 10, 24], however, the proof becomes more involved due to the appearance of additional well-formation interaction terms.

Lemma 3.4 (Pointwise lower limit)**.** *Let $0 < \beta < 1/3$. Then we have a pointwise lower limit on $Q(\rho, k)$ of the form*

$$Q(\rho, k)(x, t) \geq C_6, \qquad \forall (x, t) \in [0, 1] \times [0, T], \tag{68}$$

where the constant $C_6 = C_6(C_2, C_3, C_5, \inf_{[0,1]} Q_0, \sup_{[0,1]} Q_0, T, \|u_0\|_{L^2(I)}, \|c_0\|_{L^\gamma(I)})$.

Proof. We first define

$$v(x, t) = \frac{1}{Q(x, t)}, \qquad V(t) = \max_{[0,1]\times[0,t]} v(x, s).$$

We calculate as follows:

$$\begin{aligned} v(x, t) - v(0, t) &= \int_0^x \partial_x v \, dx \leq \int_0^1 |\partial_x Q| v^2 \, dx = \frac{1}{\beta} \int_0^1 v^{\beta+1} |\partial_x Q^\beta| \, dx \\ &\leq \frac{1}{\beta} \Big(\int_0^1 |\partial_x Q^\beta|^2 \, dx \Big)^{1/2} \Big(\int_0^1 v^{2(\beta+1)} \, dx \Big)^{1/2} \\ &\leq \frac{C_5^{1/2}}{\beta} \Big(\int_0^1 v \, dx \Big)^{1/2} \Big((\max_{[0,1]} v(\cdot, t))^{2\beta+1} \Big)^{1/2} \\ &\leq \frac{C_5^{1/2}}{\beta} \Big(\int_0^1 v \, dx \Big)^{1/2} \Big(\max_{[0,1]} v(\cdot, t) \Big)^{\beta+1/2}, \end{aligned} \tag{69}$$

where we have used (59). Next, we focus on how to estimate $\int_0^1 v \, dx$. The starting point is the observation that the second equation of (41) can be written as

$$v_t - \rho_l u_x = -[cA]v.$$

Integrating over $[0, 1] \times [0, t]$ we get

$$\begin{aligned} \int_0^1 v(x, t) \, dx &= \int_0^1 v(x, 0) \, dx + \rho_l \int_0^t [u(1, s) - u(0, s)] \, ds - \int_0^t \int_0^1 [cA] v \, dx \, ds \\ &\leq (\inf_{[0,1]} Q_0)^{-1} + 2\rho_l \int_0^t \max_{[0,1]} |u(\cdot, s)| \, ds + M \int_0^t \int_0^1 v \, dx \, ds \\ &\leq (\inf_{[0,1]} Q_0)^{-1} + 2\rho_l \sqrt{t} \Big(\int_0^t \|u^2(s)\|_{L^\infty(I)} \, ds \Big)^{1/2} + M \int_0^t \int_0^1 v \, dx \, ds, \end{aligned} \tag{70}$$

where we have used Hölder's inequality. In light of Sobolev's inequality $\|f\|_{L^\infty(I)} \le C\|f\|_{W^{1,1}(I)}$ it follows that the second last term of (70) can be estimated as follows:

$$\begin{aligned}
\int_0^t \|u^2(s)\|_{L^\infty(I)}\,ds &\le C\int_0^t \|u^2(s)\|_{W^{1,1}(I)}\,ds \\
&= C\Big(\int_0^t\int_0^1 u^2\,dx\,ds + \int_0^t\int_0^1 |(u^2)_x|\,dx\,ds\Big) \\
&\le CtC_1 + 2C\int_0^t\int_0^1 Q^{\frac{1+\beta}{2}}|u||u_x|v^{\frac{1+\beta}{2}}\,ds\,ds \qquad (71)\\
&\le CtC_1 + 2C\Big(\int_0^t\int_0^1 Q^{1+\beta}u_x^2u^2\,dx\,ds\Big)^{1/2}\Big(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\Big)^{1/2} \\
&\le CtC_1 + 2CC_3^{1/2}\Big(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\Big)^{1/2},
\end{aligned}$$

where we have used (46) and (48) with $q=2$ and Hölder's inequality. Combining (70) and (71) we get

$$\begin{aligned}
&\int_0^1 v(x,t)\,dx \\
&\le (\inf_{[0,1]} Q_0)^{-1} + 2\rho_l\sqrt{t}\Big[CtC_1 + 2CC_3^{1/2}\Big(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\Big)^{1/2}\Big]^{1/2} + M\int_0^t\int_0^1 v\,dx\,ds \\
&\le C + C\Big(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\Big)^{1/4} + M\int_0^t\int_0^1 v\,dx\,ds \qquad (72)\\
&= C + C\Big(\int_0^t\int_0^1 v^{2\beta}v^{1-\beta}\,dx\,ds\Big)^{1/4} + M\int_0^t\int_0^1 v\,dx\,ds \\
&\le C + CV(t)^{2\beta/4}\Big(\int_0^t\int_0^1 v^{1-\beta}\,dx\,ds\Big)^{1/4} + MV(t)^{\beta}\int_0^t\int_0^1 v^{1-\beta}\,dx\,ds,
\end{aligned}$$

where $C = C(\inf_{[0,1]} Q_0, C_1, T)$. Now we focus on estimating $\int_0^t\int_0^1 v^{1-\beta}\,dx\,ds$. For that purpose, we note that the second equation of (41), by multiplying with $Q^{\frac{\beta-1}{2}-1}$, can be written as

$$(Q^{\frac{\beta-1}{2}})_t = \rho_l\frac{1-\beta}{2}Q^{\frac{\beta+1}{2}}u_x - \frac{1-\beta}{2}[cA]Q^{\frac{\beta-1}{2}}.$$

Integrating this equation over $[0,t]$ we get

$$Q^{\frac{\beta-1}{2}}(x,t) = Q^{\frac{\beta-1}{2}}(x,0) + \rho_l\frac{1-\beta}{2}\int_0^t Q^{\frac{\beta+1}{2}}u_x\,ds - \frac{1-\beta}{2}\int_0^t [cA]Q^{\frac{\beta-1}{2}}\,ds.$$

Consequently, using the inequality $(a+b)^2 \le 2a^2 + 2b^2$ we get

$$\begin{aligned}
Q^{\beta-1}(x,t) &\le 2Q^{\beta-1}(x,0) + 4\rho_l^2\Big(\frac{1-\beta}{2}\Big)^2\Big(\int_0^t Q^{\frac{\beta+1}{2}}u_x\,ds\Big)^2 + 4\Big(\frac{1-\beta}{2}\Big)^2\Big(\int_0^t [cA]Q^{\frac{\beta-1}{2}}\,ds\Big)^2 \\
&\le 2Q^{\beta-1}(x,0) + \rho_l^2 t(1-\beta)^2\int_0^t Q^{\beta+1}u_x^2\,ds + M^2t(1-\beta)^2\int_0^t Q^{\beta-1}\,ds,
\end{aligned}$$

by Hölder's inequality. Integrating over $[0,1]$ in space yields

$$\begin{aligned}
&\int_0^1 v^{1-\beta}\,dx = \int_0^1 Q^{\beta-1}\,dx \\
&\le 2\int_0^1 v^{1-\beta}(x,0)\,dx + \rho_l^2t(1-\beta)^2\int_0^1\int_0^t Q^{\beta+1}u_x^2\,ds\,dx + M^2t(1-\beta)^2\int_0^1\int_0^t Q^{\beta-1}\,ds\,dx \qquad (73)\\
&\le C + M^2t(1-\beta)^2\int_0^t\int_0^1 v^{1-\beta}\,dx\,ds,
\end{aligned}$$

with $C = C(\inf_{[0,1]} Q_0, C_1, T)$ where we have used (46). Thus, by Gronwall's inequality we conclude that

$$\int_0^1 v^{1-\beta}\,dx \le C(\inf_{[0,1]} Q_0, C_1, M, T). \tag{74}$$

Consequently, (72) and (74) imply that

$$\int_0^1 v(x,t)\,dx \le C + D[V(t)^{\beta/2} + V(t)^{\beta}] \le E[1 + V(t)^{\beta/2} + V(t)^{\beta}], \tag{75}$$

for appropriate constants C, D and E that depend essentially on $\inf_{[0,1]} Q_0, M, T, C_1$. Substituting (75) into (69) we get

$$\begin{aligned}
v(x,t) - v(0,t) &\le \frac{C_5^{1/2}}{\beta}\Big(\int_0^1 v\,dx\Big)^{1/2}\Big(\max_{[0,1]} v(\cdot,t)\Big)^{\beta+1/2} \\
&\le \frac{(C_5E)^{1/2}}{\beta}[1 + V(t)^{\beta/2} + V(t)^{\beta}]^{1/2} V(t)^{\beta+1/2} \\
&\le F[1 + V(t)^{\beta/4} + V(t)^{\beta/2}] V(t)^{\beta+1/2} \\
&\le F\max(CV(t)^{(3/2)\beta+1/2}, 3),
\end{aligned} \tag{76}$$

for $F = F(C_5, E)$. Here we have used the inequality $(1+x^{\beta/4}+x^{\beta/2})x^{\beta+1/2} \le Cx^{(3/2)\beta+1/2}$ which holds for $x \ge 1$ and an appropriate constant $C \ge 3$. This follows by observing that

$$\begin{aligned}
f(x) = Cx^{(3/2)\beta+1/2} - x^{\beta+1/2}(1 + x^{\beta/4} + x^{\beta/2}) &= x^{\beta+1/2}((C-1)x^{\beta/2} - 1 - x^{\beta/4}) \\
\ge x^{\beta+1/2}((C-1)x^{\beta/2} - 1 - x^{\beta/2}) &= x^{\beta+1/2}((C-2)x^{\beta/2} - 1) \ge 0,
\end{aligned}$$

for $x \ge 1$ and $C \ge 3$.

We must check that $v(0,t)$ remains bounded in $[0,T]$. From the boundary condition (44) we have

$$c^{\gamma}Q^{\gamma} - Q^{\beta+1}u_x\Big|_{x=0} = 0.$$

Since $A(0,t) = 0$, we get that $u_xQ^2\rho_l = -Q_t$ for $x = 0$. Hence, we also get

$$y' = -K(t)y^{\gamma-\beta+1},$$

where

$$K(t) = \rho_l c(0,t)^{\gamma} = \rho_l c_0(0)^{\gamma} = K, \qquad y(t) = Q(0,t), \qquad y_0 = Q(0,0),$$

where we have used that $A(0,t) = 0$. Hence,

$$\frac{1}{\beta-\gamma}(y^{\beta-\gamma} - y_0^{\beta-\gamma}) = -Kt, \qquad \text{or} \quad y^{\beta-\gamma} = -K(\beta-\gamma)t + y_0^{\beta-\gamma}.$$

Equivalently,

$$(y^{-1})^{\gamma-\beta} = (y_0^{-1})^{\gamma-\beta}\Big(K(\gamma-\beta)y_0^{\gamma-\beta}t + 1\Big).$$

Consequently,

$$v(0,t) = v(0,0)\Big(K(\gamma-\beta)Q(0,0)^{\gamma-\beta}t + 1\Big)^{1/(\gamma-\beta)} \le C(\sup_{[0,1]} c_0, \inf_{[0,1]} Q_0, \sup_{[0,1]} Q_0, T), \qquad t \in [0,T].$$

In conclusion, from (76) we have

$$V(T) \le C + 3F\max\Big(V(T)^{(3/2)\beta+1/2}, 1\Big).$$

Since $\beta < 1/3$ we see that $(3/2)\beta + 1/2 < 1$. Therefore, it is clear from the inequality $x \le C(1+x^{\xi})$ with $0 < \xi < 1$, that $x \le G$ for some constant G. Consequently, $V(T) \le G$ where (in view of the above estimates)

$$G = G(C_2, C_3, \inf_{[0,1]} Q_0, \sup_{[0,1]} Q_0, T, \|u_0\|_{L^2(I)}, \|c_0\|_{L^{\gamma}(I)}).$$

Thus, the result (68) follows. $\square$

Now, we can directly deduce the following pointwise estimates which ensure that no transition to single-phase flow occurs.

Corollary 3.2. *There is a constant $\mu = \mu(C_2, C_6) > 0$ such that for $(x,t) \in [0,1] \times [0,T]$, we have*

$$0 < \mu \leq \rho(x,t), \qquad [1-c]\rho(x,t) \leq \rho_l - \mu < \rho_l \tag{77}$$

$$0 < \mu \inf_{x\in[0,1]}(c) \leq n(x,t) \leq \Big(\frac{\rho_l - \mu}{1 - \sup_{x\in[0,1]}(c)}\Big) \sup_{x\in[0,1]}(c) < \infty, \tag{78}$$

for $c = n/\rho$.

Proof. In view of (39) and the bounds (47) and (68) it is clear that there is a $\mu > 0$ such that (77) holds. Consequently,

$$0 < \mu \inf_{x\in[0,1]}(c) \leq n = c\rho \leq \Big(\frac{\rho_l - \mu}{1 - \sup_{x\in[0,1]}(c)}\Big) \sup_{x\in[0,1]}(c) < \infty,$$

where we have used the estimates (36) of Corollary 3.1. □

Corollary 3.3. *We have the estimates*

$$\int_0^1 (\partial_x \rho)^2 \, dx \leq C_7, \qquad \int_0^1 (\partial_x n)^2 \, dx \leq C_8, \tag{79}$$

for a constant $C_7 = C_7(C_2, C_4, C_5, C_6)$ and $C_8 = C_8(C_2, C_4, C_5, C_6)$.

Proof. It follows that

$$\partial_x Q(\rho,k)^\beta = \beta Q(\rho,k)^{\beta-1}[Q_\rho \partial_x \rho + Q_k \partial_x k] = \beta Q(\rho,k)^{\beta+1}\Big[\frac{\rho_l}{\rho^2}\partial_x \rho + \partial_x k\Big].$$

In view of this calculation and the pointwise upper and lower limits for $Q(\rho,k)$, as well as ρ, given by (47), (68), and Corollary 3.2, it follows by application of Lemma 3.2 and Lemma 3.3 that the first estimate of (79) holds. The second follows directly from the relation

$$\partial_x n = \rho \partial_x c + c \partial_x \rho, \qquad \text{since } n = c\rho,$$

and the corresponding estimate

$$\int_0^1 (\partial_x n)^2 \, dx \leq 2(\sup_{x\in[0,1]} \rho)^2 \int_0^1 (\partial_x c)^2 + 2(\sup_{x\in[0,1]} c)^2 \int_0^1 (\partial_x \rho)^2 \, dx \leq C_8,$$

where we use the first estimate of (79), Lemma 3.2 and Corollary 3.2. □

4. Proof of existence result

Now focus is on the model (20). All arguments in this section closely follow along the line of [17], however, for completeness we include the main steps. First, we introduce the Friedrichs mollifier $j_\delta(x)$. Let $\psi(x) \in C_0^\infty(\mathbb{R})$ satisfy $\psi(x) = 1$ when $|x| \leq 1/2$ and $\psi(x) = 0$ when $|x| \geq 1$, and define $\psi_\delta := \psi(x/\delta)$.

Mollifying. We extend n_0, ρ_0, u_0 to $\mathbb{R}$ by using

$$n_0(x) := \begin{cases} n_0(1), & x \in (1,\infty), \\ n_0(x), & x \in [0,1], \\ n_0(0), & x \in (-\infty, 0), \end{cases} \qquad \rho_0(x) := \begin{cases} \rho_0(1), & x \in (1,\infty), \\ \rho_0(x), & x \in [0,1], \\ \rho_0(0), & x \in (-\infty, 0), \end{cases}$$

whereas we extend $u_0(x)$ to $\mathbb{R}$ by defining it to be zero outside the interval $[0,1]$. Approximate initial data $(n_0^\delta, \rho_0^\delta, u_0^\delta)$ to (n_0, ρ_0, u_0) are now defined as follows:

$$
\begin{aligned}
n_0^\delta(x) &= (n_0 * j_\delta)(x), \qquad \rho_0^\delta(x) = (\rho_0 * j_\delta)(x),\\
u_0^\delta &= (u_0 * j_\delta)(x)[1-\psi_\delta(x)-\psi_\delta(1-x)] + (u_0 * j_\delta)(0)\psi_\delta(x) + (u_0 * j_\delta)(1)\psi_\delta(1-x)\\
&\quad + (c_0^\delta)^\gamma Q(\rho_0^\delta)^{\gamma-\beta-1}(0)\int_0^x \psi_\delta(y)\,dy - (c_0^\delta)^\gamma Q(\rho_0^\delta)^{\gamma-\beta-1}(1)\int_x^1 \psi_\delta(1-y)\,dy.
\end{aligned}
\tag{80}
$$

Then it follows that $n_0^\delta, \rho_0^\delta \in C^{1+s}[0,1]$, $u_0^\delta \in C^{2+s}[0,1]$ for any $0 < s < 1$, and $n_0^\delta, \rho_0^\delta$ and u_0^δ are compatible with the boundary conditions (23). Moreover, it follows that

$$
\begin{aligned}
|(u_0 * j_\delta)(0)|^{2q}\int_0^1 \psi_\delta^{2q}\,dx &\le C\delta\Big(\int_0^\delta u_0(x)j_\delta(x)\,dx\Big)^{2q}\\
&\le C\delta\int_0^\delta u_0^{2q}\,dx\Big(\int_0^\delta j_\delta^{2q/(2q-1)}(x)\,dx\Big)^{2q-1}\\
&\le C\int_0^\delta u_0^{2q}(x)\,dx \to 0 \quad \text{as } \delta\to 0.
\end{aligned}
$$

Similarly, it follows that $|(u_0 * j_\delta(1)|^{2q}\int_0^1 \psi_\delta^{2q}(1-x)\,dx \to 0$. Therefore, recalling the definition of $u_0^\delta(x)$ we see that as $\delta \to 0$,

$$
u_0^\delta \to u_0 \ \text{ in } L^{2q}(I). \tag{81}
$$

In addition,

$$
n_0^\delta \to n_0, \quad \rho_0^\delta \to \rho_0 \qquad \text{uniformly in } [0,1], \tag{82}
$$

as $\delta \to 0$.

Now, we consider the initial boundary value problem (20)–(24) with the initial data (n_0, ρ_0, u_0) replaced by $(n_0^\delta, \rho_0^\delta, u_0^\delta)$. For this problem standard arguments can be used (the energy estimates and the contraction mapping theorem) to obtain the existence of a unique *local* solution $(n^\delta, \rho^\delta, u^\delta)$ with $n^\delta, n_t^\delta, n_x^\delta, n_{tx}^\delta, \rho^\delta, \rho_x^\delta, \rho_t^\delta, \rho_{tx}^\delta, u^\delta, u_x^\delta, u_t^\delta, u_{xx}^\delta \in C^{\alpha,\alpha/2}([0,1]\times[0,T^*])$ for some $T^* > 0$.

In view of the estimates of Section 3.2, it follows that n^δ and ρ^δ are pointwise bounded from above and below, $(u^\delta)^q$, n_x^δ, and ρ_x^δ are bounded in $L^\infty([0,T], L^2(I))$ and u_x^δ is bounded in $L^2((0,T), L^2(I))$ for any $T > 0$. Furthermore, we can differentiate the equations in (20) and apply the energy method to derive bounds of high-order derivatives of $(n^\delta, \rho^\delta, u^\delta)$. Then the Schauder theory for linear parabolic equations can be applied to conclude that the $C^{\alpha,\alpha/2}(D_T)$-norm of $n^\delta, n_t^\delta, n_x^\delta, n_{tx}^\delta, \rho^\delta, \rho_x^\delta, \rho_t^\delta, \rho_{tx}^\delta, u^\delta, u_x^\delta, u_t^\delta, u_{xx}^\delta$ is a priori bounded. Therefore, we can continue the local solution globally in time and obtain that there exists a unique global solution $(n^\delta, \rho^\delta, u^\delta)$ of (20)–(24) with initial data $(n_0^\delta, \rho_0^\delta, u_0^\delta)$ such that for any $T > 0$, the regularity of (29) holds.

Estimates and Compactness. Clearly, in view of the estimates of Section 3 and the model itself (20), we have

$$
\begin{aligned}
&\int_0^1 (u^\delta)^{2q}(x,t)\,dx + \int_0^1 (n_x^\delta)^2(x,t)\,dx + \int_0^1 (\rho_x^\delta)^2(x,t)\,dx \le C, \qquad t\in[0,T],\ q\in\mathbb{N},\\
&\qquad 0 < \mu \le \rho^\delta(x,t) \le \Big(\frac{\rho_l - \mu}{1-\sup_{x\in[0,1]}(c)}\Big)\sup_{x\in[0,1]}(c),\\
&\qquad 0 < \mu \inf_{x\in[0,1]}(c) \le n^\delta(x,t) \le \Big(\frac{\rho_l - \mu}{1-\sup_{x\in[0,1]}(c)}\Big)\sup_{x\in[0,1]}(c), \qquad \text{for } (x,t)\in[0,1]\times[0,T],\\
&\int_0^T\int_0^1 \Big[(u_x^\delta)^2 + (n_t^\delta)^2 + (\rho_t^\delta)^2\Big](x,s)\,dx\,ds \le C,
\end{aligned}
\tag{83}
$$

where the constants $C, \mu > 0$ do not depend on δ. Note that the boundedness of ρ_t^δ (and n_t^δ) in $L^2([0,T], L^2(I))$ follows in view of the equation $\rho_t^\delta + (\rho^\delta)^2 u_x^\delta = nA$ (and $n_t^\delta + n^\delta\rho^\delta u_x^\delta = nA$), the estimates of Corollary 3.2, and the energy estimate (46) of Lemma 3.1. Hence, we can extract a subsequence of $(n^\delta, \rho^\delta, u^\delta)$, still denoted by $(n^\delta, \rho^\delta, u^\delta)$, such that as $\delta \to 0$,

$$\begin{aligned} &u^\delta \rightharpoonup u \text{ weak-* in } L^\infty([0,T], L^{2q}(I)),\\ &n^\delta \rightharpoonup n \text{ weak-* in } L^\infty([0,T], W^{1,2}(I)),\\ &\rho^\delta \rightharpoonup \rho \text{ weak-* in } L^\infty([0,T], W^{1,2}(I)),\\ &(n_t^\delta, \rho_t^\delta, u_x^\delta) \rightharpoonup (n_t, \rho_t, u_x) \text{ weakly in } L^2([0,T], L^2(I)). \end{aligned} \tag{84}$$

Next, we show that (n, ρ, u) obtained in (84) in fact is a weak solution of (20)–(24). The classical Sobolev imbedding (Morrey's inequality) $W^{1,2q}(0,1) \hookrightarrow C^{1-1/(2q)}[0,1]$ applied with $q = 1$ gives that for any $x_1, x_2 \in (0,1)$ and $t \in [0,T]$

$$|\rho^\delta(x_1,t) - \rho^\delta(x_2,t)| \le C|x_1 - x_2|^{1/2}. \tag{85}$$

To control continuity in time, in view of the sequence of imbeddings $W^{1,2}(0,1) \hookrightarrow L^\infty(0,1) \hookrightarrow L^2(0,1)$, we can apply Lions-Aubin lemma (see for example [20], Section 1.3.12) for a constant $\nu > 0$ (arbitrary small) to find a constant C_ν such that

$$\begin{aligned} \|\rho^\delta(t_1) - \rho^\delta(t_2)\|_{L^\infty(I)} &\le \nu\|\rho^\delta(t_1) - \rho^\delta(t_2)\|_{W^{1,2}(I)} + C_\nu\|\rho^\delta(t_1) - \rho^\delta(t_2)\|_{L^2(I)}\\ &\le 2\nu\|\rho^\delta(t)\|_{W^{1,2}(I)} + C_\nu|t_1 - t_2|^{1/2}\|\rho_t^\delta\|_{L^2([0,T],L^2(I))}\\ &\le C\nu + C_\nu C|t_1 - t_2|^{1/2}, \end{aligned} \tag{86}$$

where we have used (83) to derive the last two inequalities. Consequently, (85) and (86) together with the triangle inequality show that $\{\rho^\delta\}$ is equi-continuous on $D_T = [0,1] \times [0,T]$. Hence, by Arzela-Ascoli's theorem and a diagonal process for t, we can extract a subsequence of $\{\rho^\delta\}$, such that

$$\rho^\delta(x,t) \to \rho(x,t) \text{ strongly in } C^0(D_T). \tag{87}$$

The same arguments apply to n yielding

$$n^\delta(x,t) \to n(x,t) \text{ strongly in } C^0(D_T). \tag{88}$$

Clearly, ρ_t is also bounded in $L^2([0,T], L^2(I))$ and from the estimate

$$\begin{aligned} \|\rho(t_1) - \rho(t_2)\|_{L^2(I)}^2 &= \int_0^1 |\rho(t_1) - \rho(t_2)|^2\,dx = \int_0^1 \Big|\int_{t_1}^{t_2} \rho_t\,ds\Big|^2\,dx \le \int_0^1 \Big(\int_{t_1}^{t_2} |\rho_t|\,ds\Big)^2\,dx\\ &\le |t_1 - t_2| \int_0^T \int_0^1 \rho_t^2\,dx\,ds, \end{aligned}$$

where we have used Hölder's inequality, we may also conclude that

$$\rho \in C^{1/2}([0,T], L^2(I)). \tag{89}$$

Similarly, the same arguments apply to n. Thus, we conclude that the limit functions (n, ρ, u) from (84) satisfy the first two equations $n_t + n\rho u_x = nA$ and $\rho_t + \rho^2 u_x = nA$ of (28) for a.e. $x \in (0,1)$ and any $t \ge 0$. To show that the last integral equality holds, we multiply the third equation of (20) by $\phi \in C_0^\infty(D)$ with $D = [0,1] \times [0,\infty)$ and integrate over $(0,T) \times (0,1)$, followed by integration by parts with respect to x and t. Taking the limit as $\delta \to 0$, we see that (n, ρ, u) also must satisfy weakly the third equation of (28).

5. A uniqueness result

In this section we present a uniqueness result for the two-phase model (20) similar to the one presented in [10] for the gas-liquid model with $A = 0$ and the simplified momentum equation given by (3). For that purpose we need more regularity of the fluid velocity u. More precisely, for initial data $u_0 \in H^1(I)$ we have the following result.

Lemma 5.1. *Let (n, ρ, u) be a weak solution of (20)–(24) in the sense of Theorem 2.1. If $u_0 \in H^1(I)$, then*

$$u \in L^\infty([0,T], H^1(I)) \cap L^2([0,T], H^2(I)), \qquad u_t \in L^2([0,T], L^2(I)). \tag{90}$$

More precisely, the following estimate holds:

$$\|u_t\|_{L^2(D_T)} + \|u_{xx}\|_{L^2(D_T)} + \|u_x\|_{L^\infty([0,T],L^2(I))} \le C, \tag{91}$$

where the constant C depends on the quantities involved in the estimates of Lemma 3.1–3.4.

Proof. We consider the global smooth solutions $(n^\delta, \rho^\delta, u^\delta)$ described in the previous section with initial data $(n_0^\delta, \rho_0^\delta, u_0^\delta)$ which possess smoothness properties as described by (29). It follows that (see Section 3 in [17] for more details)

$$\partial_x u_0^\delta \to \partial_x u_0 \qquad \text{in } L^2(I). \tag{92}$$

For the coming calculation the superscript δ is neglected. We multiply the third equation of (41) by u_t and integrate over $[0,1]\times[0,T]$. Applying integration by parts together with the boundary condition (44) we get

$$\begin{aligned}\int_0^t\int_0^1 u_t^2\,dx\,ds - \int_0^1 [P(c,Q)u_x - E(Q)u_x^2]\,dx + \int_0^1 [P(c_0,Q_0)u_{0,x} - E(Q_0)u_{0,x}^2]\,dx \\ + \int_0^t\int_0^1 [P(c,Q) - E(Q)u_x]_t u_x\,dx\,ds + \int_0^t\int_0^1 u_t[cA]u\,dx\,ds = 0.\end{aligned} \tag{93}$$

For the first term on the second line of (93) we have

$$\begin{aligned}&[P(c,Q) - E(Q)u_x]_t u_x \\ &= \gamma[cQ]^\gamma A u_x - \rho_l\gamma c^\gamma Q^{\gamma+1}(u_x)^2 + (\beta+1)\rho_l Q^{\beta+2}(u_x)^3 - (\beta+1)[cA]Q^{\beta+1}(u_x)^2 - Q^{\beta+1}(\frac{1}{2}u_x^2)_t,\end{aligned}$$

where we have used the second equation of (41). Observing that

$$\begin{aligned}Q^{\beta+1}(\frac{1}{2}u_x^2)_t &= (\frac{1}{2}Q^{\beta+1}u_x^2)_t - \frac{1}{2}(\beta+1)Q^\beta Q_t u_x^2 = (\frac{1}{2}E(Q)u_x^2)_t + \frac{1}{2}(\beta+1)Q^\beta[\rho_l Q^2 u_x - cAQ](u_x)^2 \\ &= (\frac{1}{2}E(Q)u_x^2)_t + \frac{1}{2}\rho_l(\beta+1)Q^{\beta+2}(u_x)^3 - \frac{1}{2}(\beta+1)[cA]Q^{\beta+1}(u_x)^2,\end{aligned}$$

it follows that

$$\begin{aligned}&\int_0^t\int_0^1 [P(c,Q) - E(Q)u_x]_t u_x\,dx\,ds \\ &\quad= -\rho_l\gamma\int_0^t\int_0^1 c^\gamma Q^{\gamma+1}(u_x)^2\,dx\,ds + \frac{1}{2}(\beta+1)\rho_l\int_0^t\int_0^1 Q^{\beta+2}(u_x)^3\,dx\,ds \\ &\quad\quad+ \gamma\int_0^t\int_0^1 [cQ]^\gamma A u_x\,dx\,ds - \frac{1}{2}(\beta+1)\int_0^t\int_0^1 [cA]Q^{\beta+1}(u_x)^2\,dx\,ds \\ &\quad\quad- \frac{1}{2}\int_0^1 E(Q)u_x^2\,dx + \frac{1}{2}\int_0^1 E(Q_0)u_{0,x}^2\,dx.\end{aligned} \tag{94}$$

From (93) and (94) it follows that

$$\begin{aligned}&\int_0^t\int_0^1 u_t^2\,dx\,ds + \frac{1}{2}\int_0^1 E(Q)u_x^2\,dx \\ &= \frac{1}{2}\int_0^1 E(Q_0)u_{0,x}^2\,dx + \int_0^1 P(c,Q)u_x\,dx - \int_0^1 P(c_0,Q_0)u_{0,x}\,dx \\ &+ \rho_l\gamma\int_0^t\int_0^1 c^\gamma Q^{\gamma+1}(u_x)^2\,dx\,ds - \frac{1}{2}(\beta+1)\rho_l\int_0^t\int_0^1 Q^{\beta+2}(u_x)^3\,dx\,ds \\ &- \gamma\int_0^t\int_0^1 [cQ]^\gamma A u_x\,dx\,ds + \frac{1}{2}(\beta+1)\int_0^t\int_0^1 [cA]Q^{\beta+1}(u_x)^2\,dx\,ds - \int_0^t\int_0^1 u_t[cA]u\,dx\,ds.\end{aligned} \tag{95}$$

The second term on the right hand side of (95) can be absorbed in the second term on the left hand side by using the Cauchy inequality with ε

$$2ab \le \varepsilon a^2 + \varepsilon^{-1}b^2, \qquad a,b>0, \qquad \varepsilon>0. \tag{96}$$

Similarly, the last term on the right hand side of (95) can be absorbed in the first term on the left hand side by use of the estimates of (83), the pointwise bound on $[cA]$, and the inequality (96). By application of the estimates of (83), regularity of initial data, and regularity on the mass flow rate function $A(x,t)$ the remaining terms on the right hand side of (95) can be estimated. We then get an estimate of the form

$$\int_0^t\int_0^1 u_t^2\,dx\,ds + \int_0^1 u_x^2\,dx \le C + C\int_0^t\int_0^1 Q^{\beta+2}(u_x)^3\,dx\,ds. \tag{97}$$

The last term of (97), in view of (83), can be estimated as follows:

$$\begin{aligned}
\int_0^t\int_0^1 Q^{\beta+2}(u_x)^3\,dx\,ds &\le C\int_0^t \max_{x\in[0,1]}(Q^{1+\beta}u_x)(\cdot,s)\Big(\int_0^1 u_x^2\,dx\Big)\,ds\\
&\le C\int_0^t \max_{x\in[0,1]}|(E(Q)u_x - P(c,Q))(\cdot,s)|\Big(\int_0^1 u_x^2\,dx\Big)\,ds + C\int_0^t\int_0^1 u_x^2\,dx\,ds\\
&\le C\int_0^t\Big(\int_0^1 |(E(Q)u_x - P(c,Q))_x|\,dx\Big)\Big(\int_0^1 u_x^2\,dx\Big)\,ds + C\\
&= C\int_0^t\Big(\int_0^1 (|u_t| + |ucA|)\,dx\Big)\Big(\int_0^1 u_x^2\,dx\Big)\,ds + C\\
&\le \frac12\int_0^t\int_0^1 u_t^2\,dx\,ds + \frac12\int_0^t\int_0^1 [cAu]^2\,dx\,ds + C\int_0^t\Big(\int_0^1 u_x^2\,dx\Big)^2\,ds + C,
\end{aligned}$$

where we have used Sobolev's inequality as well as (96) to obtain the last inequality. Inserting this in (97) gives

$$\int_0^t\int_0^1 u_t^2\,dx\,ds + \int_0^1 u_x^2\,dx \le C + \int_0^t \|u_x(s)\|^2_{L^2(I)}\int_0^1 u_x^2\,dx\,ds, \qquad \forall t\in[0,T]. \tag{98}$$

Since $\int_0^T \|u_x(s)\|^2_{L^2(I)}\,ds < \infty$, in view of see (83), application of Gronwall's inequality to (98) gives the estimate

$$\int_0^t\int_0^1 u_t^2\,dx\,ds + \int_0^1 u_x^2\,dx \le C. \tag{99}$$

The last equation of (20), the estimates of (83) and the estimate (99) imply that

$$\int_0^T\int_0^1 (u_{xx})^2\,dx\,ds \le C. \tag{100}$$

Thus, (90) and (91) have been shown. □

Taking advantage of the additional regularity of Lemma 5.1 we now derive a stability result.

Lemma 5.2. *Let (n_1,ρ_1,u_1) be an arbitrary weak solution of (20)–(24), in the sense of Theorem 2.1, which also satisfies the regularity of (90). Let (n_2,ρ_2,u_2) be another weak solution subject to the same initial data. Then we have the stability estimate*

$$\begin{aligned}
&\|u_1-u_2\|^2_{L^2(I)} + \|Q(\rho_1,k)^{-1} - Q(\rho_2,k)^{-1}\|^2_{L^2(I)}\\
&\qquad \le \int_0^t C(s)\Big(\|Q(\rho_1,k)^{-1} - Q(\rho_2,k)^{-1}\|^2_{L^2(I)} + \|u_1-u_2\|^2_{L^2(I)}\Big)\,ds,
\end{aligned} \tag{101}$$

where the non-negative constant $C(s)$ satisfies $\int_0^T C(s)\,ds < \infty$.

Proof. We consider the reformulated model as expressed by (41)–(45). In view of (37) of Corollary 3.1 it follows $c_1 = c_2 := c$. In the following it will be useful to work with the quantity $v_i = 1/Q(\rho_i,k)$, $i=1,2$. We then get

$$(Q_i^\beta)_t + \rho_l\beta Q_i^{\beta+1}u_{ix} = \beta[cA]Q_i^\beta, \qquad (v_i)_t = \rho_l u_{ix} - [cA]v_i, \qquad i=1,2. \tag{102}$$

The last equation of (41) yields

$$(u_1-u_2)_t + ([cQ(\rho_1,k)]^\gamma - [cQ(\rho_2,k)]^\gamma)_x = -(u_1-u_2)[cA] + (Q(\rho_1,k)^{\beta+1}u_{1x} - Q(\rho_2,k)^{\beta+1}u_{2x})_x.$$

Multiplying by (u_1-u_2), integrating over $[0,1]$ together with integration by parts and application of boundary conditions (44) give

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 (u_1-u_2)^2\,dx\\
&\quad=\int_0^1 ([cQ(\rho_1,k)]^\gamma-[cQ(\rho_2,k)]^\gamma)(u_1-u_2)_x\,dx\\
&\qquad\quad-\int_0^1 (u_1-u_2)^2[cA]\,dx-\int_0^1 (Q(\rho_1,k)^{\beta+1}u_{1x}-Q(\rho_2,k)^{\beta+1}u_{2x})(u_1-u_2)_x\,dx\\
&\quad=\frac{1}{\rho_l}\int_0^1 c^\gamma(v_1^{-\gamma}-v_2^{-\gamma})(v_1-v_2)_t\,dx+\frac{1}{\rho_l}\int_0^1 c^\gamma(v_1^{-\gamma}-v_2^{-\gamma})(v_1-v_2)[cA]\,dx\\
&\qquad-\int_0^1 (v_1^{-(\beta+1)}-v_2^{-(\beta+1)})(u_{1x}-u_{2x})u_{2x}\,dx-\int_0^1 Q(\rho_1,k)^{\beta+1}(u_{1x}-u_{2x})^2\,dx\\
&\qquad-\int_0^1 (u_1-u_2)^2[cA]\,dx\\
&\quad\le-\frac{1}{2\rho_l}\frac{d}{dt}\int_0^1 c^\gamma a(x,t)(v_1-v_2)^2\,dx+\frac{1}{2\rho_l}\int_0^1 c^\gamma a_t(x,t)(v_1-v_2)^2\,dx\\
&\qquad+\frac{\gamma}{2}\int_0^1 c^\gamma kAa(x,t)(v_1-v_2)^2\,dx+\frac{1}{\rho_l}\int_0^1 c^\gamma(v_1^{-\gamma}-v_2^{-\gamma})(v_1-v_2)[cA]\,dx\\
&\qquad+\frac{C_0}{2}\int_0^1 (u_{1x}-u_{2x})^2\,dx+C\int_0^1 (v_1-v_2)^2(u_{2x})^2\,dx-C_0\int_0^1 (u_{1x}-u_{2x})^2\,dx\\
&\qquad+M\int_0^1 (u_1-u_2)^2\,dx,
\end{aligned}
\tag{103}
$$

for appropriate constants C_0 and C. Here we have used that

$$
a(x,t)=\frac{f(v_1)-f(v_2)}{v_1-v_2}=\int_0^1 f'(\tau(v_1-v_2)+v_2)\,d\tau=\gamma\int_0^1 \frac{1}{(\tau(v_1-v_2)+v_2)^{(\gamma+1)}}\,d\tau, \tag{104}
$$

with $f(v)=-v^{-\gamma}$, i.e. $f'(v)=\gamma v^{-(\gamma+1)}$ so that

$$
\begin{aligned}
&\int_0^1 c^\gamma(v_1^{-\gamma}-v_2^{-\gamma})(v_1-v_2)_t\,dx=-\int_0^1 c^\gamma a(x,t)(v_1-v_2)(v_1-v_2)_t\,dx\\
&\quad=-\frac{1}{2}\int_0^1 c^\gamma a(x,t)((v_1-v_2)^2)_t\,dx\\
&\quad=-\frac{1}{2}\frac{d}{dt}\int_0^1 c^\gamma a(x,t)(v_1-v_2)^2\,dx+\frac{1}{2}\int_0^1 c^\gamma a_t(x,t)(v_1-v_2)^2\,dx+\frac{\gamma}{2}\int_0^1 c^\gamma kAa(x,t)(v_1-v_2)^2\,dx.
\end{aligned}
$$

In addition, we have used that $|g(y_1)-g(y_1)|\le\max|g'(y)||y_1-y_2|$ for $g(y)=y^{-(\beta+1)}$ together with the upper and lower limits for v_i, $i=1,2$ given by (83), as well as the inequality (96). These estimates also imply that $a(x,t)$ given by (104) has a positive lower limit on $D_T=[0,1]\times[0,T]$. Moreover,

$$
a_t(x,t)=\int_0^1 f''(\tau(v_1-v_2)+v_2)(\tau(v_{1t}-v_{2t})+v_{2t})\,d\tau,
$$

so that

$$
|a_t(x,t)|\le\int_0^1 |f''(\tau(v_1-v_2)+v_2)|(\tau|v_{1t}-v_{2t}|+|v_{2t}|)\,d\tau\le C(|v_{1t}-v_{2t}|+|v_{2t}|),
$$

where C depends on lower and upper limits of v_1 and v_2. Consequently,

$$\begin{aligned}
&\frac{1}{2\rho_l}\int_0^1 c^\gamma a_t(x,t)(v_1-v_2)^2\,dx \le C\int_0^1 (|v_{1t}-v_{2t}|+|v_{2t}|)(v_1-v_2)^2\,dx\\
&\quad = C\int_0^1 |v_{1t}-v_{2t}|(v_1-v_2)^2\,dx + C\int_0^1 |v_{2t}|(v_1-v_2)^2\,dx\\
&\quad \le C\varepsilon\int_0^1 (v_{1t}-v_{2t})^2(v_1-v_2)^2\,dx + C\varepsilon^{-1}\int_0^1 (v_1-v_2)^2\,dx + C\int_0^1 |v_{2t}|(v_1-v_2)^2\,dx\\
&\quad \le \frac{C_0}{4\rho_l^2}\int_0^1 (v_{1t}-v_{2t})^2\,dx + C\int_0^1 (1+|v_{2t}|)(v_1-v_2)^2\,dx\\
&\quad = \frac{C_0}{4}\int_0^1 (u_{1x}-u_{2x})^2\,dx + \frac{C_0M^2}{4\rho_l^2}\int_0^1 (v_1-v_2)^2\,dx + C\int_0^1 (1+|v_{2t}|)(v_1-v_2)^2\,dx,
\end{aligned}$$

where we have used (96) with an appropriate choice of $\varepsilon > 0$, the upper and lower limits of v_1 and v_2, and (102). Inserting this in (103) we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 (u_1-u_2)^2\,dx + \frac{1}{2\rho_l}\frac{d}{dt}\int_0^1 c^\gamma a(x,t)(v_1-v_2)^2\,dx + \frac{C_0}{4}\int_0^1 (u_{1x}-u_{2x})^2\,dx\\
&\quad \le C\int_0^1 (1+\rho_l|u_{2x}|+Mv_2)(v_1-v_2)^2\,dx + D\int_0^1 (v_1-v_2)^2(u_{2x})^2\,dx + M\int_0^1 (u_1-u_2)^2\,dx\\
&\quad \le C\int_0^1 [(1+\rho_l|u_{2x}|)^2+Mv_2](v_1-v_2)^2\,dx + M\int_0^1 (u_1-u_2)^2\,dx,
\end{aligned} \tag{105}$$

for a suitable choice of the constant C. Integrating over $[0,t]$ we get the inequality

$$\begin{aligned}
&\int_0^1 (u_1-u_2)^2\,dx + \int_0^1 c^\gamma a(x,t)(v_1-v_2)^2\,dx + \int_0^t\int_0^1 (u_{1x}-u_{2x})^2\,dx\,ds\\
&\quad \le C\int_0^t\int_0^1 [(1+|u_{2x}|)^2+Mv_2](v_1-v_2)^2\,dx\,ds + M\int_0^t\int_0^1 (u_1-u_2)^2\,dx\,ds.
\end{aligned} \tag{106}$$

Using that $\inf_{x\in[0,1]} a(x,t) > 0$ and $\inf_{x\in[0,1]} c(x,t) > 0$ as well as the pointwise upper bound on v_2 we get

$$\begin{aligned}
&\int_0^1 (u_1-u_2)^2(x,t)\,dx + \int_0^1 (v_1-v_2)^2(x,t)\,dx\\
&\quad \le C\int_0^t\int_0^1 (1+|u_{2x}|)^2(v_1-v_2)^2(x,s)\,dx\,ds + M\int_0^t\int_0^1 (u_1-u_2)^2(x,s)\,dx\,ds\\
&\quad \le C\int_0^t \|(1+|u_{2x}|)^2\|_{L^\infty(I)}\int_0^1 (v_1-v_2)^2(x,s)\,dx\,ds + M\int_0^t\int_0^1 (u_1-u_2)^2(x,s)\,dx\,ds.
\end{aligned} \tag{107}$$

From this the estimate (101) follows. Finally, by Sobolev's imbedding theorem we have $\|f\|_{L^\infty(I)} \le C\|f\|_{W^{1,1}(I)}$ which implies that

$$\begin{aligned}
&\int_0^t \|(1+u_{2x})^2\|_{L^\infty(I)}\,ds \le C\int_0^t \|(1+u_{2x})^2\|_{W^{1,1}(I)}\,ds\\
&\quad = C\int_0^t\int_0^1 (1+u_{2x})^2\,dxds + C\int_0^t\int_0^1 |((1+u_{2x})^2)_x|\,dx\,ds\\
&\quad \le C + C\int_0^t\int_0^1 |(1+u_{2x})u_{2xx}|\,dx\,ds\\
&\quad \le C + C\Big(\int_0^t\int_0^1 (1+u_{2x})^2\,dx\,ds\Big)^{1/2}\Big(\int_0^t\int_0^1 u_{2xx}^2\,dx\,ds\Big)^{1/2} \le C,
\end{aligned}$$

since $u_2 \in L^\infty([0,T],H^1(I)) \cap L^2([0,T],H^2(I))$ (see Lemma 5.1). □

Now, we can conclude that the following uniqueness result holds.

Theorem 5.1 (Uniqueness)**.** *Under the assumptions of Theorem 2.1 and the additional regularity assumption $u_0 \in H^1(I)$, the weak solutions are unique.*

Proof. Clearly, the results of Lemma 5.1 and Lemma 5.2 hold which lead to the inequality (101). Thus, application of Gronwall's inequality to (101) yields immediately that

$$Q(\rho_1(x,t),k) = Q(\rho_2(x,t),k), \qquad u_1(x,t) = u_2(x,t) \qquad \text{a.e. } (x,t) \in D_T = [0,1] \times [0,T].$$

The fact that $Q(\rho,k)$ is monotone relatively ρ implies the desired result. □

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